

# Parametrizing nilpotent orbits in $p$ -adic symmetric spaces using Bruhat-Tits theory

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## Abstract

Let  $k$  be a field with a nontrivial discrete valuation which is complete and has perfect residue field. Let  $G$  be the group of  $k$ -rational points of a reductive, linear algebraic group  $\mathbf{G}$  equipped with an involution  $\theta$  defined over  $k$ . Let  $\mathfrak{p}$  denote the  $(-1)$ -eigenspace in the decomposition of the Lie algebra of  $G$  under the differential  $d\theta$ . If  $\mathbf{H}$  is a subgroup of  $\mathbf{G}^\theta$ , the set of  $\theta$ -fixed points, which contains the connected component of  $\mathbf{G}^\theta$ , then  $H = \mathbf{H}(k)$  acts on  $\mathfrak{p}$ , which we treat as a symmetric space. Let  $r \in \mathbb{R}$ . Under mild restrictions on  $\mathbf{G}$  and  $k$ , the set of nilpotent  $H$ -orbits in  $\mathfrak{p}$  is parametrized by equivalence classes of noticed Moy-Prasad cosets of depth  $r$  which lie in  $\mathfrak{p}$ .

## 1 Introduction

Let  $k$  be a field equipped with a nontrivial discrete valuation, and let  $\mathbf{G}$  be a reductive, linear algebraic group defined over  $k$ . Let  $\mathfrak{g}$  be the vector space of  $k$ -rational points of  $\mathrm{Lie}(\mathbf{G})$ , and let  $G = \mathbf{G}(k)$ . Consider the adjoint action of  $G$  on  $\mathfrak{g}$ . In [[9]], DeBacker gave a uniform parametrization of the set of nilpotent  $G$ -orbits in  $\mathfrak{g}$  using Bruhat-Tits theory. It is the purpose of this paper to establish a parametrization of nilpotent orbits in the context of  $p$ -adic symmetric spaces using Bruhat-Tits theory.

More precisely, let  $\theta : \mathbf{G} \rightarrow \mathbf{G}$  be a nontrivial involution defined over  $k$ . Under  $d\theta$ , the differential of  $\theta$ ,  $\mathfrak{g}$  decomposes into  $(+1)$  and  $(-1)$ -eigenspaces, which we denote  $\mathfrak{h}$  and  $\mathfrak{p}$ , respectively. Let  $\mathbf{H}$  denote a subgroup with  $(\mathbf{G}^\theta)^\circ \subset \mathbf{H} \subset \mathbf{G}^\theta$  such that  $\mathbf{H}$  is defined over  $k$ . Vust (in [[22]]) and Prasad-Yu (in [[19], Theorem 2.4]) showed that  $\mathbf{H}^\circ$  is reductive whenever  $\mathbf{G}$  is reductive. Thus, we may consider the Bruhat-Tits building of  $H = \mathbf{H}(k)$ . (Note that  $H$  preserves  $\mathfrak{p}$  under the adjoint action.) Under the assumption that the residual characteristic of  $k$  is not two, Prasad and Yu showed (in [[19], Theorem 1.9]) that we may identify  $\mathcal{B}(H)$  with the set of  $\theta$ -fixed points in  $\mathcal{B}(G)$ . This result was also proved in the case where  $\mathbf{H}$  is a classical group arising from an involution (as well as spherical buildings) in [[14], Theorem 6.7.3]. Using this identification, it makes sense to consider elements of  $\mathcal{B}(H)$  as elements lying in  $\mathcal{B}(G)$ .

In [[17]], for each  $r \in \mathbb{R}$ , Moy and Prasad associate a lattice  $\mathfrak{g}_{x,r}$  to each point  $x \in \mathcal{B}(G)$ . If  $r = 0$  and  $x \in \mathcal{B}(H)$ , then  $\theta$  acts on each Lie algebra  $V_{x,0} := \mathfrak{g}_{x,0}/\mathfrak{g}_{x,0+}$ , which then gives

a decomposition  $V_{x,0} = V_{x,0}^+ \oplus V_{x,0}^-$  into  $(+1)$  and  $(-1)$ -eigenspaces. We can then define an action of  $H$  on the set of degenerate cosets in  $V_{x,0}^-$ , meaning those cosets which contain a nilpotent element in  $\mathfrak{g}$ . Thus, in the case when  $r = 0$ , this paper provides a parametrization of nilpotent  $H$ -orbits in  $\mathfrak{p}$  in terms of equivalence classes of pairs  $(F, e)$ , where  $F$  is the set of  $\theta$ -fixed points of a  $\theta$ -stable facet of  $\mathcal{B}(G)$ , and  $e$  is a degenerate coset in  $V_{x,0}^-$ .

Ultimately, we will be interested in doing harmonic analysis on  $G/H$ , which is referred to as a  $p$ -adic symmetric space. For  $p$ -adic symmetric spaces, spherical characters play the role of characters of irreducible, admissible representations of  $G$ . In [[20], Theorem 7.11], Rader-Rallis gave a local expansion for spherical characters of irreducible class one representations of  $G$  (see [[20], Section 1]) in a neighborhood about the identity in terms of  $H$ -invariant distributions supported on  $\mathcal{N} \cap \mathfrak{p}$ , the set of nilpotent elements in  $\mathfrak{p}$ . We can (and do) identify the  $k$ -rational points of the tangent space of  $\mathbf{G}/\mathbf{H}$  at the identity with  $\mathfrak{p}$ , and this is where the nilpotent  $H$ -orbits will live. A motivation for describing a parametrization of nilpotent  $H$ -orbits in  $\mathfrak{p}$  is to establish a homogeneity result about the spherical character of an irreducible class one representation of  $G$ . The analogous homogeneity result for characters of irreducible, admissible representations of  $G$ , which occurs in harmonic analysis on  $G$ , was given in [[10], Theorem 3.5.2].

In this paper, we focus on a particular type of facet which encodes the  $H$ -orbit structure of  $\mathcal{N} \cap \mathfrak{p}$ . Suppose  $r \in \mathbb{R}$ . As in [[9], Section 3.1], we say that  $x, y \in \mathcal{B}(G)$  belong to the same *generalized  $r$ -facet*  $F'^* \subset \mathcal{B}(G)$  if  $\mathfrak{g}_{x,r} = \mathfrak{g}_{y,r}$  and  $\mathfrak{g}_{x,r+} = \mathfrak{g}_{y,r+}$ . We will only consider the  $\theta$ -fixed points of  $\theta$ -stable generalized  $r$ -facets; we call these *generalized  $(r, \theta)$ -facets*. The generalized  $(r, \theta)$ -facets form a partition of the Bruhat-Tits building of  $H$ .

In Section 4.4, for  $x \in F_\theta^*$ , a generalized  $(r, \theta)$ -facet, we attach an  $\mathfrak{f}$ -vector space  $V_{F_\theta^*} := \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$  to  $F_\theta^*$ . We call a coset  $e$  lying in  $V_{F_\theta^*}$  a Moy-Prasad coset. Such a coset is said to be degenerate if it intersects  $\mathcal{N}$ , the set of nilpotent elements in  $\mathfrak{g}$ , nontrivially. At this point, we restrict our attention to degenerate cosets in the  $(-1)$ -eigenspace of  $V_{F_\theta^*}$ , denoted  $V_{F_\theta^*}^-$ , and let  $I_r^n$  denote the set of pairs of the form  $(F_\theta^*, e)$ , with  $F_\theta^*$  a generalized  $(r, \theta)$ -facet and  $e \in V_{F_\theta^*}^-$  a degenerate coset. In Section 4.5, we define a natural equivalence relation  $\sim$  on  $I_r^n$ . To each pair  $(F_\theta^*, e) \in I_r^n$ , with some restrictions on  $\mathbf{G}$  and  $k$  described in Section 5 (which are present in the group case in [[9]]), we associate a nilpotent  $H$ -orbit  $\mathcal{O}_\theta(F_\theta^*, e)$  in  $\mathfrak{p}$ , which (in Section 6) is described as the unique nilpotent  $H$ -orbit in  $\mathfrak{p}$  of minimal dimension intersecting  $e$  nontrivially. Let  $\mathcal{O}_\theta(0)$  denote the set of nilpotent  $H$ -orbits in  $\mathfrak{p}$ . Upon restricting to a natural subset  $I_r^d$  (the *noticed* orbits of Definition 6.18) of pairs in  $I_r^n$ , we prove the following theorem:

**Theorem 1.1.** *There is a bijective correspondence between  $I_r^d/\sim$  and  $\mathcal{O}_\theta(0)$  given by the map that sends  $(F_\theta^*, e)$  to  $\mathcal{O}_\theta(F_\theta^*, e)$ .*

Any reductive, linear algebraic group  $\mathbf{J}$  defined over a local field can be thought of as a symmetric space in the following way: let  $\mathbf{G} = \mathbf{J} \times \mathbf{J}$  and define an involution  $\theta$  by  $(x, y) \mapsto (y, x)$ . Then, the diagonal  $\mathbf{H} := \{(x, x) \mid x \in \mathbf{J}\}$  occurs as the set of  $\theta$ -fixed points, and we may identify  $\mathbf{G}/\mathbf{H}$  with  $\mathbf{J}$ . This is often referred to as the group (or diagonal) case. Note that  $\mathfrak{p}$ , as defined earlier, may be identified with  $\text{Lie}(\mathbf{J})$ . In the symmetric space setting, since we are interested in  $H$ -orbits, it is convenient to use the building of  $H$  to describe a parametrization of nilpotent  $H$ -orbits in  $\mathfrak{p}$ . More specifically, using arguments similar to those in [[9]], we are able to lift the results from the group case to the symmetric space case at each step.

For our purposes, we will primarily be concerned with the case where  $\mathbf{H}$  is isotropic over  $k$ . This contrasts with the case considered when studying symmetric spaces of real Lie groups. More specifically, recall that  $H$  preserves  $\mathfrak{p}$  under the adjoint action. We say that  $X \in \mathfrak{p}$  is nilpotent whenever there exists a one-parameter subgroup  $\lambda \in \mathbf{X}_*^k(\mathbf{G})$  such that

$$\lim_{t \rightarrow 0} \lambda(t)X = 0.$$

It will be demonstrated in Remark 5.17 that, under mild restrictions on the characteristic of  $k$ , we may assume  $\lambda$  lies in  $\mathbf{X}_*^k(\mathbf{H})$ .

## 1.1 Example

We demonstrate the parametrization in the case  $r = 0$  in the following example. Let  $k = \mathbb{Q}_p$ , with  $p \neq 2$ . Let  $\mathbf{G} = \mathbf{SL}_3$ , and consider the involution  $\theta : \mathbf{SL}_3 \rightarrow \mathbf{SL}_3$  defined by

$$A \mapsto J(A^t)^{-1}J, \text{ where } J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \text{ Under } d\theta, \text{ the Lie algebra } \mathfrak{sl}_3(k) \text{ decomposes as}$$

$$\mathfrak{sl}_3(k) = \left\{ \begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} x & y & s \\ z & -2x & y \\ u & z & x \end{pmatrix} \right\}.$$

As further explained in Appendix A, representatives for the six nilpotent  $H$ -orbits in  $\mathfrak{p}$  are  $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ,  $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ , and  $\left\{ \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \bar{z} \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2 \right\}$ . The diagonal torus in the set of  $\theta$ -fixed points,  $\mathbf{H} = \mathbf{PGL}_2$ , is a maximal  $k$ -split torus  $\mathbf{T}$  which lies in the diagonal maximal  $k$ -split torus  $\mathbf{T}'$  of  $\mathbf{SL}_3$ . If

$$t = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}$$

is an element of  $\mathbf{T}'$ , define  $\alpha$  and  $\beta$  by  $\alpha(t) = ab^{-1}$  and  $\beta(t) = ab^2$ . Then,  $\{\alpha, \beta\}$  is a choice of simple roots of  $\mathbf{T}'$  in  $\mathbf{G}$  with respect to  $k$ . We let  $\check{\alpha}$  and  $\check{\beta}$  denote the associated co-roots, respectively.

Using [[19], Theorem 1.9], we are able to identify the building of  $H$  with the set of  $\theta$ -fixed points in  $\mathcal{B}(G)$ . Thus, we identify the apartment corresponding to the diagonal torus in  $\mathbf{PGL}_2$  with an affine subspace of the apartment corresponding to the diagonal torus in  $\mathbf{SL}_3$ .

In order to provide a parameterization of the nilpotent  $PGL_2$ -orbits in  $\mathfrak{p}$ , we restrict our attention to subsets of  $\mathcal{B}(H)$  which arise naturally by considering  $\theta$ -stable facets of  $\mathcal{B}(G)$ . We call a subset  $F$  in an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  a  $\theta$ -facet if  $F$  is the set of  $\theta$ -fixed points of a  $\theta$ -stable facet  $F'$  in some apartment of  $\mathcal{B}(G)$ .

The corresponding apartments are represented in Figure 1, along with the  $\theta$ -facets arising from the apartment associated to  $\mathbf{T}$ .

The  $\theta$ -facets in Figure 1 labelled  $F_1, F_2$ , and  $F_3$  are those which arise as the fixed points of facets in the closure of a fixed alcove  $C$  of  $\mathcal{A}(\mathbf{T}', \mathbb{Q}_p)$ . In particular,  $F_1$  is the vertex at the

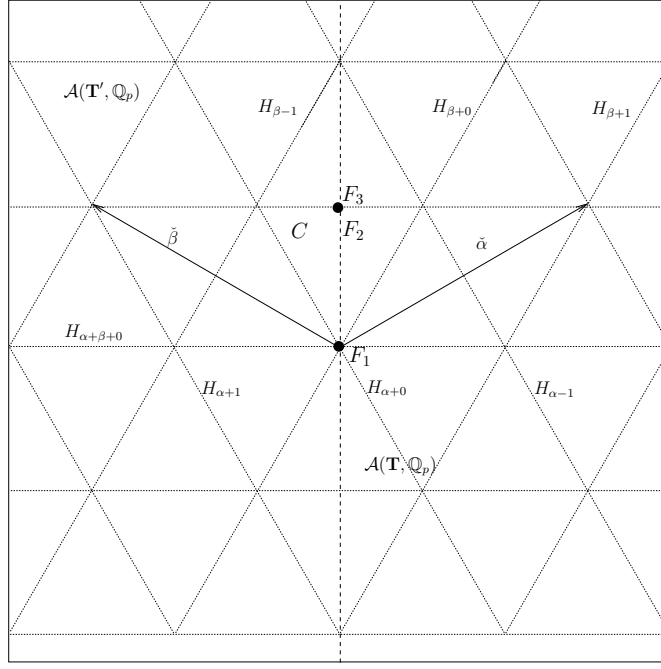


Figure 1: Affine apartments of  $G = \mathbf{SL}_3(\mathbb{Q}_p)$  and  $H = \mathbf{PGL}_2(\mathbb{Q}_p)$  (dotted line represents  $\mathcal{A}(\mathbf{T}, \mathbb{Q}_p)$ )

base of  $\overline{C}$ ,  $F_2$  is the  $\theta$ -facet arising from  $C$  itself, and  $F_3$  is the point in the closure of the alcove whose lift is the segment at the top of  $\overline{C}$ .

If  $F$  is a  $\theta$ -facet containing some point  $x \in \mathcal{B}(H)$ , we note that  $\theta$  induces a map, which we denote  $d\theta_F$ , on the Lie algebra  $V_F := \mathfrak{g}_x / \mathfrak{g}_x^+$ . In this way, we may consider the decomposition of  $V_F$  under  $d\theta_F$ , and examine nilpotent  $H$ -orbits in the  $(-1)$ -eigenspace of  $V_F$ . The corresponding Lie algebras associated to each of these  $\theta$ -facets are listed below:

$$V_{F_1} = \left\{ \begin{pmatrix} \mathbb{Z}_p/p\mathbb{Z}_p & \mathbb{Z}_p/p\mathbb{Z}_p & \mathbb{Z}_p/p\mathbb{Z}_p \\ \mathbb{Z}_p/p\mathbb{Z}_p & \mathbb{Z}_p/p\mathbb{Z}_p & \mathbb{Z}_p/p\mathbb{Z}_p \\ \mathbb{Z}_p/p\mathbb{Z}_p & \mathbb{Z}_p/p\mathbb{Z}_p & \mathbb{Z}_p/p\mathbb{Z}_p \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} x & y & s \\ z & -2x & y \\ u & z & x \end{pmatrix} \right\},$$

$$V_{F_2} = \left\{ \begin{pmatrix} \mathbb{Z}_p/p\mathbb{Z}_p & 0 & 0 \\ 0 & \mathbb{Z}_p/p\mathbb{Z}_p & 0 \\ 0 & 0 & \mathbb{Z}_p/p\mathbb{Z}_p \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & -2x & 0 \\ 0 & 0 & x \end{pmatrix} \right\},$$

and

$$V_{F_3} = \left\{ \begin{pmatrix} \mathbb{Z}_p/p\mathbb{Z}_p & 0 & p^{-1}\mathbb{Z}_p/\mathbb{Z}_p \\ 0 & \mathbb{Z}_p/p\mathbb{Z}_p & 0 \\ p\mathbb{Z}_p/p^2\mathbb{Z}_p & 0 & \mathbb{Z}_p/p\mathbb{Z}_p \end{pmatrix} \right\} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} x & 0 & p^{-1}s \\ 0 & -2x & 0 \\ pu & 0 & x \end{pmatrix} \right\},$$

with all lowercase entries being representatives in  $\mathbb{Z}_p$ .

At this point, we would like to match up nilpotent  $H$ -orbits with nilpotent orbits arising from each of the above  $\mathfrak{f}$ -Lie algebras. In order to obtain a bijection, however, we must

restrict ourselves to elements  $e \in V_F^-$  whose centralizer (in  $V_F^+$ ) does not contain certain noncentral (meaning elements in  $V_F^+$  which do not belong to the center of  $V_F$ ) semisimple elements which are fixed by  $\theta$ . This may be thought of as a restriction on the type of Levi subalgebra which is allowed to contain  $e$  and thus resembles the distinguished condition found in [[9], Remark 5.5.2]. We call such nilpotent elements *noticed*.

The noticed nilpotent  $H$ -orbits in  $V_{F_1}^-$  have representatives of the form  $\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$  and  $\left\{ \begin{pmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \bar{s} \in \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2 \right\}$ . The only noticed nilpotent  $H$ -orbit in  $V_{F_2}^-$  is the trivial orbit. There are two more noticed nilpotent  $H$ -orbits lying in  $V_{F_3}^-$  whose lifts modulo  $p^{-1}$  correspond to the two representatives of square classes in  $\mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$ . Upon taking lifts, these six orbits clearly match up with the six nilpotent  $H$ -orbits in  $\mathfrak{p}$  discussed above.

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## 2 Preliminaries

### 2.1 Algebraic groups, involutions, and associated notation

Let  $k$  be a field with a nontrivial discrete valuation  $\nu$ , and let  $K$  be a maximal unramified extension of  $k$ . Let  $R$  (resp.  $R_K$ ) denote the ring of integers in  $k$  (resp.  $K$ ), and let  $\mathfrak{f}$  (resp.  $\mathfrak{F}$ ) denote the residue field of  $k$  (resp.  $K$ ). We assume  $k$  is complete and that  $\mathfrak{f}$  is perfect. Let  $\mathbf{G}$  be a reductive, linear algebraic group defined over  $k$ , and fix an involution  $\theta$  of  $\mathbf{G}$  which is defined over  $k$ . Let  $\mathbf{G}^\circ$  denote the connected component of  $\mathbf{G}$ .

We fix a uniformizer  $\varpi$  of  $k$ , with respect to  $\nu$ , and let  $L$  denote the minimal Galois extension of  $K$  over which  $\mathbf{G}^\circ$  splits. Let  $\ell = [L : K]$ . If  $\nu$  also denotes the extension of  $\nu$  to  $L$ , then we normalize  $\nu$  so that  $\nu(L^\times) = \mathbb{Z}$ .

If  $k'$  is any field, we let  $\bar{k}'$  denote an algebraic closure of  $k'$ . Suppose  $\mathbf{C}$  is a linear algebraic group defined over  $k'$ . We will identify  $\mathbf{C}$  with the  $\bar{k}'$ -points of  $\mathbf{C}$ . If  $\sigma$  is an involution of  $\mathbf{C}$  defined over  $k'$ , we will almost always let  $\mathbf{C}^\sigma$  denote the set  $\{x \in \mathbf{C} \mid \sigma(x) = x\}$ ; in the case that  $\mathbf{C}$  is a  $\sigma$ -stable torus, we will let  $\mathbf{C}^\sigma$  be the connected component of this set. Lastly, if  $C$  is any group, we let  $[C, C]$  denote its derived subgroup, and if  $L$  is any Lie algebra, we let  $[L, L]$  denote its derived subalgebra.

Let  $\mathbf{H}$  be a subgroup of  $\mathbf{G}$  with  $(\mathbf{G}^\theta)^\circ \subset \mathbf{H} \subset \mathbf{G}^\theta$  such that  $\mathbf{H}$  is defined over  $k$ . By the second paragraph in [[22], Section 1.0],  $\mathbf{H}^\circ$  is reductive. We let  $G$  denote the group of  $k$ -rational points of  $\mathbf{G}$  and similarly let  $H$  denote the group of  $k$ -rational points of  $\mathbf{H}$ . The involution  $\theta$  induces an involution, which we denote  $d\theta$ , on the Lie algebra,  $\mathfrak{g} := \text{Lie}(\mathbf{G})$ , of  $\mathbf{G}$ . Under  $d\theta$ ,  $\mathfrak{g}$  decomposes as  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  where  $\mathfrak{h}$  is the Lie algebra of  $\mathbf{H}$  and  $\mathfrak{p}$  is the  $(-1)$ -eigenspace of  $\mathfrak{g}$ . We let  $\mathfrak{g} = \mathfrak{g}(k)$ ,  $\mathfrak{h} = \mathfrak{h}(k)$  and  $\mathfrak{p} = \mathfrak{p}(k)$  denote the vector spaces of  $k$ -rational points of  $\mathfrak{g}$ ,  $\mathfrak{h}$ , and  $\mathfrak{p}$ . If  $V$  is a  $k'$ -vector space on which some  $k'$ -involution  $\sigma$  acts, we let  $V^+$  and  $V^-$  denote the sets  $\{X \in V \mid \sigma(X) = X\}$  and  $\{X \in V \mid \sigma(X) = -X\}$ ,

respectively.

We follow some notational conventions (also found in [[9], 2.1]) so that when we refer to a *Levi subgroup* of  $\mathbf{G}$  (resp.  $\mathbf{H}$ ), we mean a Levi subgroup of  $\mathbf{G}^\circ$  (resp.  $\mathbf{H}^\circ$ ). We apply the same terminology to tori and parabolic subgroups.

Let  $\text{Ad}$  denote the adjoint action of  $G$  on  $\mathfrak{g}$ . For  $g \in G$  and  $X \in \mathfrak{g}$ , let  ${}^gX = \text{Ad}(g)(X)$ . Suppose  $\mathbf{L}$  is a linear algebraic group defined over  $k'$  acting on its Lie algebra  $\mathfrak{l}$  via the adjoint action. If  $g \in \mathbf{L}$ , we will let  $\text{Int}(g)$  denote conjugation by  $g$ . Let  $L \supset J$  and  $\mathfrak{j}$  be subsets of  $\mathbf{L}$  and  $\mathfrak{l}$  respectively. Then, let  $C_L(\mathfrak{j}) = \{g \in L \mid {}^gX = X \text{ for all } X \in \mathfrak{j}\}$ . Similarly, we let  $N_L(\mathfrak{j}) = \{g \in L \mid {}^g\mathfrak{j} = \mathfrak{j}\}$ .

Although we restrict ourselves to discussing nilpotent elements lying in  $\mathfrak{p}$ , our definition of nilpotence will be as in [[9]]. In particular, we call  $X \in \mathfrak{p}$  nilpotent provided there exists some  $\lambda \in \mathbf{X}_*^k(\mathbf{G})$  such that  $\lim_{t \rightarrow 0} \lambda(t)X = 0$ . We let  $\mathcal{N}$  denote the set of nilpotent elements in  $\mathfrak{g}$  and define  $\mathcal{N}^- := \mathcal{N} \cap \mathfrak{p}$ . We also let  $\mathcal{U}$  denote the set of unipotent elements in  $G$ .

If the residue field  $\mathfrak{f}$  has positive characteristic, we denote the characteristic of  $\mathfrak{f}$  by  $p$ . If the residue field has characteristic zero, we let  $p = \infty$ .

## 2.2 The Bruhat-Tits building, apartments, and $\theta$ -fixed points

We let  $\mathcal{B}(G)$  denote the (enlarged) Bruhat-Tits building of  $\mathbf{G}^\circ(k)$ , and, similarly, let  $\mathcal{B}(H)$  denote the (enlarged) Bruhat-Tits building of  $\mathbf{H}^\circ(k)$ . Unless otherwise stated, the symbol  $\mathcal{B}(G)$  (resp.  $\mathcal{B}(H)$ ) will always refer to the enlarged Bruhat-Tits building; that is, it takes the center of  $\mathbf{G}^\circ(k)$  (resp.  $\mathbf{H}^\circ(k)$ ) into account. We note that since  $K/k$  is a maximal unramified extension,  $\mathfrak{F}$  is an algebraic closure of  $\mathfrak{f}$ , and  $\mathcal{B}(G)$  can be identified with the  $\text{Gal}(K/k)$ -fixed points of  $\mathcal{B}(\mathbf{G}, K)$ , the Bruhat-Tits building of  $\mathbf{G}^\circ(K)$ .

If  $\mathbf{S}$  (resp.  $\mathbf{S}'$ ) is a maximal  $k$ -split torus of  $\mathbf{H}$  (resp.  $\mathbf{G}$ ), we will let  $\mathcal{A}(\mathbf{S}, k)$  (resp.  $\mathcal{A}(\mathbf{S}', k)$ ) denote the associated apartment in  $\mathcal{B}(H)$  (resp.  $\mathcal{B}(G)$ ). If  $\mathcal{A}$  is an apartment in  $\mathcal{B}(H)$ , and  $\Omega$  is a subset of  $\mathcal{A}$ , we let  $A(\Omega, \mathcal{A})$  denote the smallest affine subspace of  $\mathcal{A}$  containing  $\Omega$ . We let  $\text{dist}: \mathcal{B}(G) \times \mathcal{B}(G) \rightarrow \mathbb{R}_+$  denote a nontrivial  $G$ -invariant distance function as described in [[21], 2.3]. For  $x, y \in \mathcal{B}(G)$ , let  $[x, y]$  denote the geodesic in  $\mathcal{B}(G)$  from  $x$  to  $y$  with respect to  $\text{dist}$ . Let  $(x, y)$  denote  $[x, y] \setminus \{x\}$ .

For  $x \in \mathcal{B}(\mathbf{G}, K)$ , we let  $\mathbf{G}(K)_x$  and  $\mathbf{G}(K)_x^+$  denote the parahoric subgroup associated to  $x$  and its pro-unipotent radical, respectively. The groups  $\mathbf{G}(K)_x$  and  $\mathbf{G}(K)_x^+$  only depend on the facet in  $\mathcal{B}(\mathbf{G}, K)$  containing  $x$ . Therefore, if  $F \subset \mathcal{B}(\mathbf{G}, K)$  is the facet containing  $x$ , we define  $\mathbf{G}(K)_F$  and  $\mathbf{G}(K)_F^+$  to be  $\mathbf{G}(K)_x$  and  $\mathbf{G}(K)_x^+$ , respectively. The quotient  $\mathbf{G}(K)_F / \mathbf{G}(K)_F^+$  is the group of  $\mathfrak{F}$ -points of a connected, reductive group  $\mathbf{G}_F$  defined over  $\mathfrak{f}$ .

If  $x \in \mathcal{B}(G)$ , we denote the parahoric associated to  $x$  and its pro-unipotent radical by  $G_x$  and  $G_x^+$ , respectively. We recall that these subgroups are obtained as the sets of  $\text{Gal}(K/k)$ -fixed points of parahorics defined over the maximal unramified extension. That is, we have  $G_F = \mathbf{G}(K)_F^{\text{Gal}(K/k)}$  and  $G_F^+ = (\mathbf{G}(K)_F^+)^{\text{Gal}(K/k)}$ . The quotient  $G_x / G_x^+$  coincides with the group of  $\mathfrak{f}$ -rational points of the connected, reductive group  $\mathbf{G}_x$  defined over  $\mathfrak{f}$ . Moreover, we have  $\mathbf{G}_x(\mathfrak{f}) = \mathbf{G}_x^{\text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})}$ .

If  $\mathbf{S}$  (resp.  $\mathbf{S}'$ ) is a maximal  $k$ -split torus in  $\mathbf{H}$  (resp.  $\mathbf{G}$ ), we let  $\Phi = \Phi(\mathbf{S}, k)$  (resp.  $\Phi(\mathbf{S}', k)$ ) denote the set of roots of  $\mathbf{S}$  (resp.  $\mathbf{S}'$ ) in  $\mathbf{H}$  (resp.  $\mathbf{G}$ ) with respect to  $k$ . If  $\mathcal{A}$  (resp.  $\mathcal{A}'$ ) is the apartment corresponding to  $\mathbf{S}$  (resp.  $\mathbf{S}'$ ), let  $\Psi = \Psi(\mathcal{A})$  (resp.  $\Psi(\mathcal{A}')$ ) denote the set of affine roots of  $\mathbf{H}$  (resp.  $\mathbf{G}$ ) with respect to  $\mathbf{S}$ ,  $k$  and  $\nu$  (resp.  $\mathbf{S}'$ ,  $k$  and  $\nu$ ). If  $\psi \in \Psi$  is an affine root, we let  $\dot{\psi} \in \Phi$  denote the gradient of  $\psi$ . Whenever  $\psi$  is an affine root of  $\mathbf{H}$  (resp.  $\mathbf{G}$ ) with

respect to  $\mathbf{S}$  (resp.  $\mathbf{S}'$ ) and  $\Omega$  is a subset of the apartment associated to  $\mathbf{S}$  (resp.  $\mathbf{S}'$ ), we let  $\text{res}_\Omega \psi$  denote the restriction of  $\psi$  to  $\Omega$ .

Throughout this paper, we will assume that  $p \neq 2$ . Under this assumption, Prasad and Yu proved that  $\mathcal{B}(H) = \mathcal{B}(G^\theta)$  can be identified with  $\mathcal{B}(G)^\theta$ . We will abuse notation and let  $\theta$  also denote the induced map on the building of  $G$ . Whenever  $\Omega'$  is a subset of  $\mathcal{B}(G)$ , we will define  $\Omega'^\theta := \{x \in \Omega' \mid \theta(x) = x\}$ .

## 2.3 The Moy-Prasad filtrations

Let  $x \in \mathcal{B}(H)$  and  $r \in \mathbb{R}$ . We let  $\mathfrak{g}_{x,r}$  denote the Moy-Prasad filtration of depth  $r$  as defined in [[17], Section 3.2]. As we will show in Section 4, the filtration lattices  $\mathfrak{g}_{x,r}$  and  $\mathfrak{g}_{x,r+}$  are  $\theta$ -stable and thus induce an  $\mathfrak{f}$ -involution of the  $\mathfrak{f}$ -vector space  $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r+}$ . We will denote this involution  $d\theta_x$ .

Let  $\mathbf{S}$  be a maximal  $k$ -split torus in  $\mathbf{H}$ , and suppose  $\mathbf{T}$  is a maximal  $K$ -split  $k$ -torus in  $\mathbf{H}$  containing  $\mathbf{S}$ . By [[19], Theorem 1.9], we can choose a maximal  $k$ -split torus  $\mathbf{S}'$  of  $\mathbf{G}$  containing  $\mathbf{S}$  and a maximal  $K$ -split  $k$ -torus  $\mathbf{T}'$  of  $\mathbf{G}$  containing  $\mathbf{S}'$  and  $\mathbf{T}$ . Since  $\mathbf{G}$  is quasi-split over  $K$ , we know that  $\mathbf{Z}' := C_{\mathbf{G}^\circ}(\mathbf{T}')$  is a maximal  $k$ -torus in  $\mathbf{G}$  containing  $\mathbf{T}'$ . We will define  $\mathbf{Z}$  to be the  $C_{\mathbf{H}^\circ}(\mathbf{T})$ , which is a maximal  $k$ -torus of  $\mathbf{H}$ .

Let  $\mathfrak{z}'$  denote the Lie algebra of  $\mathbf{Z}'$ . Following [[17], 3.2], there is a filtration of  $\mathfrak{z}'(K)$  for each  $r \in \mathbb{R}$  which we denote  $\mathfrak{z}'(K)_r$ . Moreover, for each affine functional  $\psi \in \Psi(\mathcal{A}(\mathbf{T}', K))$ , there exists a lattice denoted  $\mathfrak{u}_\psi$ , which lies in the root space in  $\mathfrak{g}(K)$  with respect to  $\mathbf{T}, \mathbf{G}$ , and  $\psi$ . The lattice  $\mathfrak{g}(K)_{x,r}$  is defined as the the  $R_K$ -submodule of  $\mathfrak{g}(K)$  spanned by  $\mathfrak{z}'(K)_r$  and the  $\mathfrak{u}_\psi$ 's for which  $\psi(x) \geq r$ .

## 3 $\theta$ -stable $k$ -split tori

### 3.1 A result of Prasad-Yu

In order to discuss a type of facet which takes both  $\theta$  and the facet structure of  $\mathcal{B}(G)$  into account, it will be useful to discuss the relationship between apartments in  $\mathcal{B}(H)$  and those in  $\mathcal{B}(G)$ . Prasad and Yu have shown in [[19], Theorem 1.9] that there is an  $\mathbf{H}^\circ(k)$ -equivariant map  $\iota : \mathcal{B}(H) \rightarrow \mathcal{B}(G)$  such that the image is  $\mathcal{B}(G)^\theta$ , uniquely defined up to translation by  $\mathbf{X}_*(\mathbf{C}) \otimes \mathbb{R}$ , where  $\mathbf{C}$  is the maximal  $k$ -split torus in the center of  $\mathbf{H}$ . Moreover, for every maximal  $k$ -split torus  $\mathbf{S}$  of  $\mathbf{H}$ , there is a maximal  $k$ -split torus  $\mathbf{S}'$  of  $\mathbf{G}$  such that  $\mathcal{A}(\mathbf{S}, k)$  is mapped into  $\mathcal{A}(\mathbf{S}', k)$  by an affine transformation. In [[19], Lemma 1.9.3], it is shown that such a map is compatible with unramified base change. In particular, there is an  $\mathbf{H}^\circ(K)$ -equivariant map  $\iota_K : \mathcal{B}(\mathbf{H}, K) \rightarrow \mathcal{B}(\mathbf{G}, K)$  which is also  $\text{Gal}(K/k)$ -equivariant, such that the restriction of  $\iota_K$  to  $\mathcal{B}(H)$  shares the same properties as  $\iota$ .

A proof of the next proposition is given in [[19], Remark 1.5.4] when  $k'$  is a completion of  $K$  and in [[14], Corollary 5.7(i)] when  $k'$  is a  $p$ -adic field or [[14], Proposition 3.4.1] when  $k'$  is a finite field of odd characteristic. In the statement of the following proposition, let  $k'$  be one of the fields mentioned above, and suppose  $\mathbf{G}'$  is a reductive linear algebraic  $k'$ -group equipped with a  $k'$ -involution  $\sigma$  with  $\mathbf{H}' = (\mathbf{G}^\sigma)^\circ$ .

**Proposition 3.1.** *Let  $\mathbf{S}$  be a maximal  $k'$ -split torus of  $\mathbf{H}'$ . Then there exists a  $\sigma$ -stable maximal  $k'$ -split torus of  $\mathbf{G}'$  which contains  $\mathbf{S}$ .*

Now, let  $k$  be as in Section 2.1. Suppose  $\mathbf{S}$  is a maximal  $k$ -split torus of  $\mathbf{H}$ . The group defined by  $\mathbf{M} := C_{\mathbf{G}^\circ}(\mathbf{S})$  is a reductive group defined over  $k$ . Moreover,  $\mathbf{M}$  is  $\theta$ -stable since  $\mathbf{S}$  is a  $\theta$ -stable torus. Thus, applying [[12], Proposition 2.3] to  $\mathbf{M}$ , there exists a  $\theta$ -stable maximal  $k$ -torus  $\mathbf{Z}$  such that the maximal  $k$ -split torus  $\mathbf{T}$  in  $\mathbf{Z}$  is a maximal  $k$ -split torus in  $\mathbf{M}$ . In particular, the torus  $\mathbf{T}$  is a  $\theta$ -stable maximal  $k$ -split torus of  $\mathbf{G}$  which contains  $\mathbf{S}$ .

**Remark 3.2.** *Let  $\mathbf{S}'$  be a  $\theta$ -stable maximal  $k$ -split torus of  $\mathbf{G}$ . The condition for  $\mathbf{S}'^\theta$  to be a maximal  $k$ -split torus of  $\mathbf{H}$  is that  $\mathbf{S}'$  lies in a minimal  $\theta$ -stable parabolic  $k$ -subgroup of  $\mathbf{G}$ . ([12], Prop. 4.5)) Under some restrictions on  $k$  and the derived subgroup of  $\mathbf{G}$ , existence of a  $\theta$ -stable  $k$ -parabolic subgroup is shown in [[12], Proposition 4.4].*

**Remark 3.3.** *It is not true, in general, that a  $\theta$ -stable apartment of  $\mathcal{B}(G)$  gives rise to an apartment in  $\mathcal{B}(H)$ . Consider  $\mathbf{SL}_2$  and the involution  $\theta$  defined by  $X \mapsto (X^t)^{-1}$ . Supposing  $-1 \in (\mathbb{Q}_p^\times)^2$ , the fixed points under this involution consist of a maximal  $k$ -split torus. However, the diagonal torus is a  $\theta$ -stable maximal  $k$ -split torus in  $\mathbf{G}$  whose set of  $\theta$ -fixed points is  $\{\pm 1\}$ .*

## 4 Equivalence of facets

### 4.1 $(r, \theta)$ -facets and Moy-Prasad lattices

Fix an apartment  $\mathcal{A}'$  of  $\mathcal{B}(G)$ . For  $\psi \in \Psi(\mathcal{A}')$ , define the hyperplane

$$H_{\psi-r} := \{x \in \mathcal{A}' \mid \psi(x) = r\}.$$

As in [[9], Section 3.1], we call a nonempty subset  $F' \subset \mathcal{A}'$  an  $r$ -facet of  $\mathcal{A}'$  if there is some finite subset  $S \subset \Psi(\mathcal{A}')$  for which

1.  $F' \subset H_S := \bigcap_{\psi \in S} H_{\psi-r}$
2.  $F'$  is a connected component (in  $H_S$ ) of  $H_S \setminus \bigcup_{\psi \in \Psi(\mathcal{A}') \setminus S} (H_S \cap H_{\psi-r})$ .

If  $F'$  is an  $r$ -facet in  $\mathcal{A}'$ , we define its dimension to be the dimension  $A(F', \mathcal{A}')$ .

The following remark, which is a consequence of the definitions above, will be important for later discussion of  $\theta$ -stable  $r$ -facets.

**Remark 4.1.** *If  $F'_1, F'_2$  are  $r$ -facets in  $\mathcal{A}'$ , and  $F'_2 \cap A(F'_1, \mathcal{A}') \neq \emptyset$ , then  $F'_2$  is entirely contained in  $A(F'_1, \mathcal{A}')$ . To see why this is true, write*

$$A(F'_1, \mathcal{A}') = H_S := \bigcap_{\psi \in S} H_{\psi-r},$$

where  $S$  is finite and  $F'_1$  is a connected component of  $H_S \setminus \bigcup_{\psi \in \Psi(\mathcal{A}') \setminus S} (H_S \cap H_{\psi-r})$ . Write

$$A(F'_2, \mathcal{A}') = H_{S'} := \bigcap_{\rho \in S'} H_{\rho-r},$$

where  $S'$  is finite and  $F'_2$  is a connected component of  $H_{S'} \setminus \bigcup_{\rho \in \Psi(\mathcal{A}') \setminus S'} (H_{S'} \cap H_{\rho-r})$ .

It will be enough to show that  $S \subset S'$ . If  $S$  is empty, then the statement is obviously true, so suppose  $\psi \in S \setminus S'$ . Let  $x_2 \in F'_2 \cap A(F'_1, \mathcal{A}')$ . Then, since  $F'_2$  lies in  $H_{S'} \setminus \bigcup_{\rho \in \Psi(\mathcal{A}') \setminus S'} (H_{S'} \cap H_{\rho-r})$ , we must have  $x_2 \in \mathcal{A}' \setminus H_{\psi-r}$ . But, since  $x_2 \in A(F'_1, \mathcal{A}')$ , we have  $\psi(x_2) = r$ , a contradiction. Thus, we must have  $S \subset S'$ , that is,  $F'_2$  is entirely contained in  $A(F'_1, \mathcal{A}')$ .



**Definition 4.2.** Define an  $(r, \theta)$ -facet to be a nonempty subset  $F$  in an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  such that there exists an apartment  $\mathcal{A}' \subset \mathcal{B}(G)$  and an  $r$ -facet  $F' \subset \mathcal{A}'$  with  $F = F'^\theta$ .

**Remark 4.3.** In Definition 4.2, we have defined a structure on apartments in  $\mathcal{B}(H)$  which is finer than the  $r$ -facet structure of apartments in  $\mathcal{B}(H)$ . For example, take  $r = 0$ ,  $\mathbf{G} = \mathbf{SL}_2$  equipped with the involution  $\theta(A) = J(A^t)^{-1}J$  as in Example 1.1. From Figure 1, we see that  $\theta$ -facet  $F_2$  is a strictly smaller subset of the  $H$ -alcove of  $\mathcal{B}(H)$  that has boundary  $\{F_1, F_1 + (\check{\alpha} + \check{\beta})\}$

**Definition 4.4.** Let  $F$  be an  $(r, \theta)$ -facet in an apartment  $\mathcal{A} \subset \mathcal{B}(H)$ . Define the dimension of an  $(r, \theta)$ -facet  $F$  by

$$\dim F := \dim A(F, \mathcal{A})$$

Let  $\mathcal{A}'$  be an apartment of  $\mathcal{B}(G)$ . Suppose  $F$  is an  $(r, \theta)$ -facet which lies inside an  $r$ -facet  $F' \subset \mathcal{A}'$ , and let  $x, y \in F$ . Then, in particular, since  $x$  and  $y$  lie inside the  $r$ -facet  $F'$ , we have  $\mathfrak{g}_{x,r} = \mathfrak{g}_{y,r}$  and  $\mathfrak{g}_{x,r^+} = \mathfrak{g}_{y,r^+}$ . A proof of this statement can be found in [[9], 3.1.4]. This allows us to make the following definition.

**Definition 4.5.** Let  $F$  be an  $(r, \theta)$ -facet of  $\mathcal{A}$ . Fix  $x \in F$ . Set

$$\mathfrak{g}_F := \mathfrak{g}_{x,r}$$

and

$$\mathfrak{g}_F^+ := \mathfrak{g}_{x,r^+}.$$

**Lemma 4.6.** Let  $F$  be an  $(r, \theta)$ -facet of  $\mathcal{A} \subset \mathcal{B}(H)$ . Suppose  $x \in \mathcal{A}$ . Then  $x \in F$  if and only if  $\mathfrak{g}_{x,r} = \mathfrak{g}_F$  and  $\mathfrak{g}_{x,r^+} = \mathfrak{g}_F^+$ .

*Proof.* We apply the analogous result in [[9], 3.1.4]. If  $x \in F$ , then, by definition, we have  $\mathfrak{g}_{x,r} = \mathfrak{g}_F$  and  $\mathfrak{g}_{x,r^+} = \mathfrak{g}_F^+$ . Suppose  $F = F'^\theta$ , where  $F'$  is an  $r$ -facet in an apartment  $\mathcal{A}' \subset \mathcal{B}(G)$ . Then, by [[9], Lemma 3.1.4], since  $\mathfrak{g}_{x,r} = \mathfrak{g}_F = \mathfrak{g}_{F'}$  and  $\mathfrak{g}_{x,r^+} = \mathfrak{g}_F^+ = \mathfrak{g}_{F'}^+$ , we have  $x \in F' \cap \mathcal{A} = F$ . □

Let  $x \in \mathcal{B}(H)$ . Before showing that there is a reasonable decomposition of the Moy-Prasad lattice  $\mathfrak{g}_{x,r}$  with respect to  $\theta$ , we demonstrate the relationship between the Moy-Prasad lattices  $\mathfrak{h}_{x,r}$  and  $\mathfrak{g}_{x,r}$ . The statements we make when discussing these lattices make sense because of [[19], Theorem 1.9].

We first make an observation. Consider the parahoric subgroups  $\mathbf{H}(K)_x$  and  $\mathbf{G}(K)_x$ . It is clear that  $\mathbf{H}(K)_x \subset \text{stab}_{\mathbf{G}(K)}(x)$ . By the argument given in [[19], Prop 1.7], we must have  $\mathbf{H}(K)_x \subset \mathbf{G}(K)_x \cap \mathbf{H}(K)$ . On the other hand, the map  $\theta$  induces an involution of the smooth, affine  $R$ -group scheme associated to  $\mathbf{G}(K)_x$ , which we denote  $\mathcal{G}$ . Call the fixed points of this group scheme  $\mathcal{G}^\theta$ . Since  $\mathbf{H}(K)_x \subset \mathbf{G}(K)_x^\theta$ , we have an induced inclusion of  $\mathcal{H} \subset \mathcal{G}^\theta$ , where  $\mathcal{H}$  is the smooth, connected, affine  $R$ -group scheme associated to the parahoric  $\mathbf{H}(K)_x$ . Let  $\mathcal{G}'$  be the smooth, affine (not necessarily connected)  $R$ -group scheme associated to  $\text{stab}_{\mathbf{H}(K)}(x)$ . Then, we have inclusions

$$\mathcal{H} \subset \mathcal{G}^\theta \subset \mathcal{G}',$$

where the group scheme  $\mathcal{H}$  is of finite index in  $\mathcal{G}'$ . Taking Lie algebras and  $R_K$ -rational points, since  $x$  is  $\text{Gal}(K/k)$ -fixed, this gives us the equality  $\mathfrak{h}_x = \mathfrak{g}_x \cap \mathfrak{h}$ .

**Lemma 4.7.** *Let  $x \in \mathcal{B}(H)$ . Then, we have*

$$\mathfrak{g}_{x,r} \cap \mathfrak{h} = \mathfrak{h}_{x,r}.$$

*Proof.* For this proof only, if  $M/k$  is a finite extension, let  $\text{ord}(M)$  denote the image  $\nu(M^\times)$ , where  $\nu$  denotes the extension of  $\nu$  to  $M$ . Following the proof of [[24], Lemma 8.2], we will first assume that  $r \in \text{ord}(k)$ . If  $\pi_r$  is an element of  $k$  with valuation  $-r$ , then  $\pi_r \mathfrak{g}_{x,r} = \mathfrak{g}_{x,0} = \mathfrak{g}_x$ , so the result follows from the observation preceding this proof.

If  $r \in \text{ord}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ , we use [[1], 1.4.1] to reduce the statement to the one above. The statement of the proposition now follows by noting that for any real number  $r$ , we have

$$\mathfrak{g}_{x,r} = \bigcap_{s < r, s \in \mathbb{Q}} \mathfrak{g}_{x,s}.$$

□

**Proposition 4.8.** *Assume  $p \neq 2$ . Let  $F_i$  ( $i=1,2$ ) be nonempty  $(r, \theta)$ -facets in some apartment  $\mathcal{A} \subset \mathcal{B}(H)$ , and suppose  $F'_i \supset F_i$  is an  $r$ -facet in some apartment  $\mathcal{A}' \subset \mathcal{B}(G)$  containing  $\mathcal{A}$ . Define  $\mathfrak{p}_{F_i} := \mathfrak{p} \cap \mathfrak{g}_{F_i}$  and  $\mathfrak{p}_{F_i}^+ := \mathfrak{p} \cap \mathfrak{g}_{F_i}^+$ . Then, for  $i = 1, 2$ , we have:*

1.  $\mathfrak{g}_{F_i} = \mathfrak{h}_{F_i} \oplus \mathfrak{p}_{F_i}$  and  $\mathfrak{g}_{F_i}^+ = \mathfrak{h}_{F_i}^+ \oplus \mathfrak{p}_{F_i}^+$
2.  $F'_1 = F'_2$  if and only if  $F_1 = F_2$

*Proof.* Since  $\theta$  is an automorphism of  $\mathbf{G}$  defined over  $k$ , it induces an action on  $\mathcal{B}(G)$  which is compatible with all structures on  $\mathcal{B}(G)$ . In particular, by [[2], Proposition 2.2.1], since  $F_i \subset \mathcal{B}(H)$ , we have that  $\mathfrak{g}_{F_i}$  and  $\mathfrak{g}_{F_i}^+$  are  $\theta$ -stable. Let  $X \in \mathfrak{g}_{F_i} = \mathfrak{g}_{x,r}$ . Since  $\mathfrak{g}_{x,r}$  is  $\theta$ -stable, and  $p \neq 2$ , we write

$$X = \frac{X + \theta(X)}{2} + \frac{X - \theta(X)}{2} \in (\mathfrak{h} \cap \mathfrak{g}_{x,r}) \oplus (\mathfrak{p} \cap \mathfrak{g}_{x,r}).$$

Since  $\mathfrak{h}, \mathfrak{p}$  and  $\mathfrak{g}_{x,r}$  and  $\mathfrak{g}_{x,r^+}$  are all  $R$ -modules, by Lemma 4.7, the above computation shows

$$\mathfrak{g}_{x,r} = (\mathfrak{h} \cap \mathfrak{g}_{x,r}) \oplus (\mathfrak{p} \cap \mathfrak{g}_{x,r}) = \mathfrak{h}_{F_i} \oplus \mathfrak{p}_{F_i}$$

and

$$\mathfrak{g}_{x,r^+} = (\mathfrak{h} \cap \mathfrak{g}_{x,r^+}) \oplus (\mathfrak{p} \cap \mathfrak{g}_{x,r^+}) = \mathfrak{h}_{F_i}^+ \oplus \mathfrak{p}_{F_i}^+.$$

For the second claim, the forward implication is trivial. For the other direction, note that if  $F_1 = F_2$ , then

$$\mathfrak{g}_{F'_1} = \mathfrak{g}_{F_1} = \mathfrak{g}_{F_2} = \mathfrak{g}_{F'_2}.$$

and

$$\mathfrak{g}_{F'_1}^+ = \mathfrak{g}_{F_1}^+ = \mathfrak{g}_{F_2}^+ = \mathfrak{g}_{F'_2}^+$$

which is true if and only if  $F'_1 = F'_2$  by [[9], Lemma 3.1.4].

□

## 4.2 Generalized $(r, \theta)$ -facets

For the following definition, recall (from [[9], 3.2.1]) that for  $x \in \mathcal{B}(G)$ , the set  $F^*(x) = \{y \in \mathcal{B}(G) \mid \mathfrak{g}_{x,r} = \mathfrak{g}_{y,r} \text{ and } \mathfrak{g}_{x,r^+} = \mathfrak{g}_{y,r^+}\}$  is called a generalized  $r$ -facet.

**Definition 4.9.** Let  $x \in \mathcal{B}(H)$ . Define

$$F_\theta^*(x) := F^*(x)^\theta.$$

**Definition 4.10.**

$$\mathcal{F}_\theta(r) := \{F_\theta^*(x) \mid x \in \mathcal{B}(H)\}.$$

We call an element of  $\mathcal{F}_\theta(r)$  a generalized  $(r, \theta)$ -facet.

**Remark 4.11.** We briefly mention a fact which will be used many times throughout this section. By [[5], 4.6.28], if  $\mathcal{A}, \tilde{\mathcal{A}}$  are two apartments in  $\mathcal{B}(H)$  such that  $\Omega = \{x, y\} \subset \mathcal{A} \cap \tilde{\mathcal{A}}$ , then there exists an element  $h \in H_\Omega$  such that  $h\mathcal{A} = \tilde{\mathcal{A}}$ . More succinctly stated,  $H_\Omega$  acts transitively on the apartments of  $\mathcal{B}(H)$  containing  $\Omega$ .

**Remark 4.12.** We remark that if  $F_\theta^*(x)$  is a generalized  $(r, \theta)$ -facet, and  $\mathcal{A}$  is an apartment of  $\mathcal{B}(H)$  such that  $F_\theta^*(x) \cap \mathcal{A} \neq \emptyset$ , then  $F_\theta^*(x) \cap \mathcal{A}$  is an  $(r, \theta)$ -facet of  $\mathcal{A}$ .

**Lemma 4.13.** Let  $x \in \mathcal{B}(H)$  and  $\mathcal{A}$  an apartment in  $\mathcal{B}(H)$  such that  $F := F_\theta^*(x) \cap \mathcal{A} \neq \emptyset$ . For all  $y \in F$ , we have

$$F_\theta^*(x) = H_y \cdot F.$$

*Proof.* Let  $y \in F$ .

“ $\subset$ ”: Let  $z \in F_\theta^*(x)$ , and let  $\tilde{\mathcal{A}}$  be an apartment in  $\mathcal{B}(H)$  containing  $y$  and  $z$ . By Remark 4.11, there exists an element  $h \in H_y$  such that  $hz \in \mathcal{A}$ . We have

$$\mathfrak{g}_{hz,r} = {}^h\mathfrak{g}_{z,r} = {}^h\mathfrak{g}_{x,r} = {}^h\mathfrak{g}_{y,r} = \mathfrak{g}_{hy,r} = \mathfrak{g}_{y,r} = \mathfrak{g}_{x,r}$$

and similarly,

$$\mathfrak{g}_{hz,r^+} = {}^h\mathfrak{g}_{z,r^+} = {}^h\mathfrak{g}_{x,r^+} = {}^h\mathfrak{g}_{y,r^+} = \mathfrak{g}_{hy,r^+} = \mathfrak{g}_{y,r^+} = \mathfrak{g}_{x,r^+}.$$

Thus,  $hz \in \mathcal{A} \cap F_\theta^*(x) = F$ , so, in particular,  $z \in H_y \cdot F$ .

“ $\supset$ ”: Let  $z \in F$  and  $h \in H_y$ . Then

$$\mathfrak{g}_{hz,r} = {}^h\mathfrak{g}_{z,r} = {}^h\mathfrak{g}_{y,r} = \mathfrak{g}_{hy,r} = \mathfrak{g}_{y,r} = \mathfrak{g}_{x,r}.$$

and

$$\mathfrak{g}_{hz,r^+} = {}^h\mathfrak{g}_{z,r^+} = {}^h\mathfrak{g}_{y,r^+} = \mathfrak{g}_{hy,r^+} = \mathfrak{g}_{y,r^+} = \mathfrak{g}_{x,r^+}.$$

Thus,  $hz \in F_\theta^*(x)$ . □

**Corollary 4.14.** If  $F_\theta^* \in \mathcal{F}_\theta(r)$ , then the image of  $F_\theta^*$  in the reduced building of  $H$  is bounded.

*Proof.* Let  $x \in F_\theta^*$ , and let  $F^*$  be the generalized  $r$ -facet in  $\mathcal{B}(G)$  containing  $x$ . The result follows directly from the result in [[9], Corollary 3.2.7] since the image of  $F_\theta^*$  is contained in the image of  $F^*$  in the reduced building.  $\square$

**Lemma 4.15.** *Let  $x \in B(H)$ . We have*

$$N_H(\mathfrak{g}_{x,r}) \cap N_H(\mathfrak{g}_{x,r^+}) = \text{stab}_H(F_\theta^*(x)).$$

*Proof.* “ $\supset$ ”: If  $h \in \text{stab}_H(F_\theta^*(x))$ , then  $hx \in F_\theta^*(x)$ , so, in particular,

$${}^h\mathfrak{g}_{x,r} = \mathfrak{g}_{hx,r} = \mathfrak{g}_{x,r}$$

and

$${}^h\mathfrak{g}_{x,r^+} = \mathfrak{g}_{hx,r^+} = \mathfrak{g}_{x,r^+}.$$

“ $\subset$ ”: Let  $n \in N_H(\mathfrak{g}_{x,r}) \cap N_H(\mathfrak{g}_{x,r^+})$ . Choose  $z \in F_\theta^*(x)$ . Since  $n$  normalizes the lattices  $\mathfrak{g}_{x,r}$  and  $\mathfrak{g}_{x,r^+}$ , we have

$$\mathfrak{g}_{nz,r} = {}^n\mathfrak{g}_{z,r} = {}^n\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r}$$

and

$$\mathfrak{g}_{nz,r^+} = {}^n\mathfrak{g}_{z,r^+} = {}^n\mathfrak{g}_{x,r^+} = \mathfrak{g}_{x,r^+}.$$

This implies, by definition of  $F^*(x)$ , that  $nz \in F^*(x)$ . Since  $z \in \mathcal{B}(H)$ , and  $n \in H$ , we have  $nz \in F^*(x) \cap \mathcal{B}(H) = F^*(x)^\theta = F_\theta^*(x)$ . Thus, since  $z \in F_\theta^*(x)$  was arbitrary,  $nF_\theta^*(x) \subset F_\theta^*(x)$  and thus  $n \in \text{stab}_H(F_\theta^*(x))$ .  $\square$

**Lemma 4.16.** *Let  $F_\theta^* \in \mathcal{F}_\theta(r)$  and  $\mathcal{A}$  an apartment in  $B(H)$  such that  $F := F_\theta^* \cap \mathcal{A} \neq \emptyset$ . Then*

$$\overline{F} = \overline{F_\theta^*} \cap \mathcal{A}.$$

*Proof.* “ $\subset$ ”: This inclusion is clear since  $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$  for any two subsets  $A, B$  of  $\mathcal{B}(H)$ . “ $\supset$ ”: Let  $x \in \overline{F_\theta^*} \cap \mathcal{A}$ . We will produce a sequence converging to  $x$  which lies in  $F$ . Since  $x \in \overline{F_\theta^*}$ , there exists a sequence  $\{x_n\}$  in  $F_\theta^*$  converging to  $x$ . Fix  $y \in F$ . Without loss of generality, assume  $\text{dist}(x_n, x) < \frac{1}{n}$  for each  $n$ . Note that  $\bigcup_{x \in \overline{C}} \overline{C}$  contains a neighborhood of  $x$ , where  $C$  ranges over all alcoves in  $\mathcal{B}(H)$ . Thus, for large  $n$ , there exist alcoves  $C_n \subset \mathcal{B}(H)$  such that  $x_n$  and  $x$  lie in  $\overline{C_n}$ . Let  $\mathcal{A}_n$  be an apartment in  $\mathcal{B}(H)$  which contains  $C_n$  and  $y$ . We now fix  $n$ . Since  $x$  and  $y$  lie in  $\mathcal{A}_n \cap \mathcal{A}$ , by Remark 4.11, there is an element  $h_n \in H$  that maps  $\mathcal{A}_n$  to  $\mathcal{A}$  and fixes  $x$  and  $y$ . In particular, since  $h_n x = x$ , we have

$$\text{dist}(h_n x_n, x) = \text{dist}(x_n, x) < \frac{1}{n}.$$

Now, by Lemma 4.15, since  $h_n y = y$  for each  $n$ , we have  $h_n \in N_H(\mathfrak{g}_{y,r}) \cap N_H(\mathfrak{g}_{y,r^+}) = \text{stab}_H(F_\theta^*)$ . Thus,  $h_n x_n \in F_\theta^*$  and since  $h_n \mathcal{A}_n = \mathcal{A}$ , we also have  $h_n x_n \in \mathcal{A}$ , so  $h_n x_n \in F_\theta^* \cap \mathcal{A} = F$ . Therefore,  $\{h_n x_n\}$  is our desired sequence.  $\square$

**Definition 4.17.** *For  $F_\theta^* \in \mathcal{F}_\theta(r)$  and  $\delta > 0$ , define*

$$F_\theta^*(\delta) := \{x \in F_\theta^* \mid \text{dist}(x, z) \geq \delta \text{ for all } z \in \overline{F_\theta^*} \setminus F_\theta^*\}.$$

**Lemma 4.18.** *Suppose  $F_\theta^* \in \mathcal{F}_\theta(r)$  and  $\delta > 0$ . Then  $F_\theta^*(\delta)$  is a convex, closed,  $\text{stab}_H(F_\theta^*)$ -invariant subset of  $\mathcal{B}(H)$ . Also,  $F_\theta^*(\delta)$  is nonempty if and only if there exists an apartment  $\mathcal{A}$  in  $\mathcal{B}(H)$  such that the following subset of  $F := F_\theta^* \cap \mathcal{A}$*

$$F_{\theta, \mathcal{A}}(\delta) := \{x \in F \mid \text{dist}(x, z) \geq \delta \text{ for all } z \in \overline{F} \setminus F\}$$

is nonempty.

*Proof.* To see that  $F_\theta^*(\delta)$  is closed, suppose  $\{x_n\} \subset F_\theta^*(\delta)$  is a sequence converging to some  $x \in \mathcal{B}(H)$ . By the triangle inequality, we have

$$\text{dist}(x, z) \geq \text{dist}(x_n, z) - \text{dist}(x_n, x) \geq \delta - \text{dist}(x_n, x),$$

for all  $z \in \overline{F_\theta^*} \setminus F_\theta^*$ , so taking  $n \rightarrow \infty$ , we see that  $\text{dist}(x, z) \geq \delta$ .

For  $\text{stab}_H(F_\theta^*)$ -invariance, we note that any element  $h \in \text{stab}_H(F_\theta^*)$  sends the boundary of  $F_\theta^*$  to itself. In particular, for  $z \in \overline{F_\theta^*} \setminus F_\theta^*$ , there exists an element  $w \in \overline{F_\theta^*} \setminus F_\theta^*$  such that  $hw = z$ . Thus, if  $x \in F_\theta^*(\delta)$ , then

$$(\dagger) \text{dist}(hx, z) = \text{dist}(hx, hw) = \text{dist}(x, w) \geq \delta.$$

We now prove the final statement of the lemma.

“ $\Rightarrow$ ” : If  $F_\theta^*(\delta)$  is nonempty, then there exists some apartment  $\mathcal{A}$  for which  $F_\theta^*(\delta) \cap \mathcal{A}$ , and hence  $F_{\theta, \mathcal{A}}(\delta)$ , is nonempty.

“ $\Leftarrow$ ” : We will prove a stronger claim here that will be used later to prove convexity of  $F_\theta^*(\delta)$ . In particular, we show that if there is an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  such that  $F_{\theta, \mathcal{A}}(\delta) \neq \emptyset$ , then  $H_y F_{\theta, \mathcal{A}}(\delta) = F_\theta^*(\delta)$  for all  $y \in F_{\theta, \mathcal{A}}(\delta)$ . This will show that  $F_\theta^*(\delta) \neq \emptyset$  whenever  $F_{\theta, \mathcal{A}}(\delta) \neq \emptyset$ . Suppose  $\mathcal{A} \subset \mathcal{B}(H)$  is an apartment such that  $F_{\theta, \mathcal{A}}(\delta) \neq \emptyset$ , and let  $w \in F_{\theta, \mathcal{A}}(\delta)$ . We first show  $H_w F_{\theta, \mathcal{A}}(\delta) \subset F_\theta^*(\delta)$ .

“ $\subset$ ” : Note that  $H_w \subset \text{stab}_H(F_\theta^*)$  by an application of Lemma 4.15. Thus by  $(\dagger)$ , we have that  $F_\theta^*(\delta)$  is  $H_w$ -invariant. As a consequence, it suffices to show that  $F_{\theta, \mathcal{A}}(\delta) \subset F_\theta^*(\delta)$ . Let  $x \in F_{\theta, \mathcal{A}}(\delta)$  and  $z \in \overline{F_\theta^*} \setminus F_\theta^*$ . By [[4], 2.3.1], we may choose an apartment  $\tilde{\mathcal{A}}$  containing  $x$  and  $z$ . By Remark 4.11, there is an element  $h \in H_x$  that maps  $\tilde{\mathcal{A}}$  onto  $\mathcal{A}$ . Since  $hx = x$ , by an application of Lemma 4.15,  $h \in \text{stab}_H(F_\theta^*)$ . Since  $h\tilde{\mathcal{A}} = \mathcal{A}$ , we have  $hz \in \mathcal{A}$ , so in particular,  $hz \in (\overline{F_\theta^*} \setminus F_\theta^*) \cap \mathcal{A}$ . Thus, by Lemma 4.16,

$$hz \in (\overline{F_\theta^*} \cap \mathcal{A}) \setminus (F_\theta^* \cap \mathcal{A}) = \overline{F_{\theta, \mathcal{A}}} \setminus F_{\theta, \mathcal{A}}.$$

Again, since  $h \in H_x$ , we have  $\text{dist}(x, z) = \text{dist}(x, hz) \geq \delta$ , so since  $z$  was arbitrary, we must have  $x \in F_\theta^*(\delta)$ .

“ $\supset$ ” : Now, we show that  $H_w F_{\theta, \mathcal{A}}(\delta) \supset F_\theta^*(\delta)$  for all  $w \in F_{\theta, \mathcal{A}}(\delta)$ . Let  $x \in F_\theta^*(\delta)$ . By Lemma 4.13, there exist elements  $h \in H_w$  and  $z \in F$  such that  $hz = x$ . Arguing as usual, since  $w \in F_\theta^*$ , we have  $H_w \subset \text{stab}_H(F_\theta^*)$ , so, in particular,

$$x \in hF \cap F_\theta^*(\delta) = h(F \cap F_\theta^*(\delta)) \subset hF_{\theta, \mathcal{A}}(\delta).$$

Lastly, we must show that  $F_\theta^*(\delta)$  is a convex subset of  $\mathcal{B}(H)$ . Assume  $F_\theta^*(\delta)$  is nonempty. We first show that  $F_{\theta,\mathcal{A}}(\delta) \subset \mathcal{A}$  is convex. Choose an origin  $O$  in  $\mathcal{A}$ . Note that the geodesics of  $\mathcal{A}$  are nothing more than segments, so if  $x, y \in F_{\theta,\mathcal{A}}(\delta)$  and  $z \in [x, y]$ , then considering  $x, y$ , and  $z$  as the vectors,  $x - O, y - O$ , and  $z - O$ , respectively, we have  $z = tx + (1 - t)y$  for some  $t \in [0, 1]$ . Thus, for all  $z' \in \overline{F} \setminus F$ ,

$$\text{dist}(tx + (1 - t)y, z') \geq \text{dist}(tx, tz') - \text{dist}((1 - t)y, (1 - t)z') = t\text{dist}(x, z') + (1 - t)\text{dist}(y, z') \geq \delta.$$

so  $z$  lies in  $F_{\theta,\mathcal{A}}(\delta)$ .

Now, suppose  $F_\theta^*(\delta)$  is nonempty. There is an apartment  $\mathcal{A}$  such that  $F_{\theta,\mathcal{A}}(\delta)$  is nonempty. Let  $x, y \in F_\theta^*(\delta)$  and  $z \in F_{\theta,\mathcal{A}}(\delta)$ . Previously in this proof, we showed that for all  $w \in F_{\theta,\mathcal{A}}(\delta)$ , we have

$$(\dagger) \quad H_w F_{\theta,\mathcal{A}}(\delta) = F_\theta^*(\delta),$$

so there is some  $h \in H_z$  such that  $hx \in F_{\theta,\mathcal{A}}(\delta)$ . Since  $h \in H_z \subset \text{stab}_H(F_\theta^*)$ , we have  $hy \in F_\theta^*$ , so for all  $z' \in \overline{F_\theta^*} \setminus F_\theta^*$ , we have

$$\text{dist}(hy, z') = \text{dist}(hy, hw') = \text{dist}(y, w') \geq \delta$$

where  $w' \in \overline{F_\theta^*} \setminus F_\theta^*$  such that  $hw' = z$ . Thus,  $hy \in F_\theta^*(\delta)$ . Applying  $(\dagger)$  again, there exists some  $h' \in H_{hx}$  such that  $h'hy \in F_{\theta,\mathcal{A}}(\delta)$ . Thus, since  $F_{\theta,\mathcal{A}}^*(\delta)$  is convex, we have

$$[h'hx, h'hy] \subset F_{\theta,\mathcal{A}}^*(\delta) \subset F_\theta^*(\delta)$$

so, in particular, since  $F_{\theta,\mathcal{A}}^*(\delta)$  is  $\text{stab}_H(F_\theta^*)$ -invariant,  $[x, y] \subset F_\theta^*(\delta)$ . □

**Definition 4.19.** For  $F_\theta^* \in \mathcal{F}_\theta(r)$ , define

$$C(F_\theta^*) := \{y \in F_\theta^* \mid \text{for all apartments } \mathcal{A} \text{ of } \mathcal{B}(H) \text{ for which } \mathcal{A} \cap F_\theta^* \neq \emptyset \text{ we have } y \in \mathcal{A}\}.$$

**Remark 4.20.** Suppose  $H$  is semisimple. Following the discussion in [[21], 2.2.1], there is a map

$$\pi : H \times \mathcal{B}(H) \rightarrow \mathcal{B}(H) \times \mathcal{B}(H)$$

given by  $(h, x) \mapsto (hx, x)$ , with the property that the inverse images of bounded sets are bounded. We note that if  $\Omega$  is a bounded subset of  $\mathcal{B}(H)$ , this tells us that

$$\text{stab}_H(\Omega) \times \Omega = \pi^{-1}(\Omega \times \Omega)$$

is bounded. In particular,  $\text{stab}_H(\Omega)$  is bounded whenever  $\Omega$  is bounded.

**Corollary 4.21.** If  $F_\theta^* \in \mathcal{F}_\theta(r)$ , then  $C(F_\theta^*) \neq \emptyset$ .

*Proof.* Let  $\text{pr}_{ss}$  denote the projection from the enlarged building of  $H$  to the reduced building of  $H$ . Since

$$C(\text{pr}_{ss}(F_\theta^*)) \neq \emptyset \Rightarrow C(F_\theta^*) \neq \emptyset,$$

we may assume  $\mathbf{H}$  is semisimple.

By Corollary 4.14,  $F_\theta^*$  is bounded in  $\mathcal{B}(H)$ , so, by Remark 4.20, the stabilizer  $N := \text{stab}_H(F_\theta^*)$  is a bounded subgroup of  $H$ . If  $F_\theta^*$  consists of a point  $x$ , then  $\mathcal{A} \cap F_\theta^* = \tilde{\mathcal{A}} \cap F_\theta^* = \{x\}$  for all apartments in  $\mathcal{B}(H)$  that meet  $\{x\}$ , so clearly  $x \in C(F_\theta^*)$ .

Suppose  $F_\theta^*$  is not a point, and let  $\mathcal{A} \subset \mathcal{B}(H)$  be an apartment such that  $F_\theta^* \cap \mathcal{A} \neq \emptyset$ . Then, by Lemma 4.13, since  $\dim F_\theta^* > 0$ , we have  $\dim F_\theta^* \cap \mathcal{A} > 0$ . As a consequence, there exists some  $\delta > 0$  for which  $F_{\theta, \mathcal{A}}(\delta)$  is nonempty. By Lemma 4.18, this implies  $\emptyset \neq F_\theta^*(\delta)$  is convex and  $N$ -stable. Thus, by [[5], 3.2.4], since a bounded group of isometries acting on a nonempty, closed, convex set  $F$  of  $\mathcal{B}(H)$  has a fixed point, there exists some  $y \in F_\theta^*(\delta)$  such that  $ny = y$  for all  $n \in N$ . Suppose now that  $\tilde{\mathcal{A}}$  is an apartment of  $\mathcal{B}(H)$  for which  $F_{\tilde{\mathcal{A}}} := F_\theta^* \cap \tilde{\mathcal{A}} \neq \emptyset$ . Let  $z \in F_{\tilde{\mathcal{A}}}$ . By Lemma 4.13, we have  $H_z F_{\tilde{\mathcal{A}}} = F_\theta^*$ , and by Lemma 4.15, we have  $H_z \subset N$ . In particular,  $hy = y$  for all  $h \in H_z$ . Therefore,  $y \in F_{\tilde{\mathcal{A}}} \subset \tilde{\mathcal{A}}$ , so  $y \in C(F_\theta^*)$ .  $\square$

**Corollary 4.22.** *If  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are two apartments in  $\mathcal{B}(H)$ , and  $F_\theta^* \in \mathcal{F}_\theta(r)$  such that  $F_\theta^* \cap \mathcal{A}_i \neq \emptyset$ , then  $\dim A(F_\theta^* \cap \mathcal{A}_1, \mathcal{A}_1) = \dim A(F_\theta^* \cap \mathcal{A}_2, \mathcal{A}_2)$ .*

**Definition 4.23.** *Let  $F_\theta^* \in \mathcal{F}_\theta(r)$ . Let  $\mathcal{A}$  be an apartment in  $\mathcal{B}(H)$  with  $F_\theta^* \cap \mathcal{A} \neq \emptyset$ . Define*

$$\dim F_\theta^* := \dim A(F_\theta^* \cap \mathcal{A}, \mathcal{A}).$$

**Definition 4.24.** *Suppose  $F_\theta^* \in \mathcal{F}_\theta(r)$ . Fix  $x \in F_\theta^*$  and define*

$$\mathfrak{g}_{F_\theta^*} := \mathfrak{g}_{x,r}$$

and

$$\mathfrak{g}_{F_\theta^*}^+ := \mathfrak{g}_{x,r^+}.$$

Let  $F_\theta^*$  be a generalized  $(r, \theta)$ -facet. From this point on, we will refer to the lattice  $\mathfrak{g}_{F_\theta^*}^+$  frequently. Recall that the quotient  $V := \mathfrak{g}_{F_\theta^*} / \mathfrak{g}_{F_\theta^*}^+$  is an  $\mathfrak{f}$ -vector space. Moreover, by Proposition 4.8, both  $\mathfrak{g}_{F_\theta^*}$  and  $\mathfrak{g}_{F_\theta^*}^+$  are  $\theta$ -stable, so  $V$  is equipped with an involution induced by  $\theta$ . We will abuse notation and call this induced map  $\theta$ . In order to avoid confusion in the next lemma, we will let  $V^{\theta-1}$  denote the set of  $\theta$ -fixed points in  $V$  and let  $V^{\theta+1} = \{v \in V \mid \theta(v) = -v\}$ . We reserve the symbol  $+$  for the lattice  $\mathfrak{g}_{F_\theta^*}^+$ .

**Lemma 4.25.** *Assume  $p \neq 2$ . Suppose  $F_\theta^*$  is a generalized  $(r, \theta)$ -facet of  $\mathcal{B}(H)$ . Then*

$$\mathfrak{g}_{F_\theta^*} / \mathfrak{g}_{F_\theta^*}^+ = \mathfrak{h}_{F_\theta^*} / \mathfrak{h}_{F_\theta^*}^+ \oplus \mathfrak{p}_{F_\theta^*} / \mathfrak{p}_{F_\theta^*}^+,$$

where we use Lemma 4.7 to identify  $\mathfrak{h}_{F_\theta^*} / \mathfrak{h}_{F_\theta^*}^+$  and  $\mathfrak{p}_{F_\theta^*} / \mathfrak{p}_{F_\theta^*}^+$  with their images inside  $\mathfrak{g}_{F_\theta^*} / \mathfrak{g}_{F_\theta^*}^+$ .

*Proof.* We first show that  $\mathfrak{h}_{F_\theta^*}/\mathfrak{h}_{F_\theta^*}^+$  is the  $(+1)$ -eigenspace of  $\mathfrak{g}_{F_\theta^*}/\mathfrak{g}_{F_\theta^*}^+$  under  $\theta$ . By Lemma 4.7, we have  $\mathfrak{h}_{x,r} = \mathfrak{g}_{x,r} \cap \mathfrak{h}$ . Thus, we can identify  $\mathfrak{h}_{F_\theta^*}/\mathfrak{h}_{F_\theta^*}^+$  as a subset of  $\mathfrak{g}_{F_\theta^*}/\mathfrak{g}_{F_\theta^*}^+$ . Moreover, it is clear that  $\theta$  fixes every element of  $\mathfrak{h}_{F_\theta^*}/\mathfrak{h}_{F_\theta^*}^+$ . Thus,  $\mathfrak{h}_{F_\theta^*}/\mathfrak{h}_{F_\theta^*}^+ \subset (\mathfrak{g}_{F_\theta^*}/\mathfrak{g}_{F_\theta^*}^+)^{\theta-1}$ .

Now, let  $\overline{X} \in (\mathfrak{g}_{F_\theta^*}/\mathfrak{g}_{F_\theta^*}^+)^{\theta-1}$ , and let  $X$  be a lift of  $\overline{X}$  in  $\mathfrak{g}_{F_\theta^*}$ . By Proposition 4.8, we have  $\mathfrak{g}_{F_\theta^*} = \mathfrak{h}_{F_\theta^*} \oplus \mathfrak{p}_{F_\theta^*}$ , so we may write  $X = X_+ + X_-$ , where  $\theta(X_+) = X_+$  and  $\theta(X_-) = -X_-$ , with  $X_+ \in \mathfrak{h}_{F_\theta^*}$  and  $X_- \in \mathfrak{p}_{F_\theta^*}$ . Note that  $\theta(X) - X \in \mathfrak{g}_{F_\theta^*}^+$ , so we have

$$-2X_- = \theta(X_+ + X_-) - (X_+ + X_-) \in \mathfrak{g}_{F_\theta^*}^+.$$

Thus, since  $p \neq 2$ , we may conclude that  $X_- \in \mathfrak{g}_{F_\theta^*}^+$ . Thus  $X_+ + \mathfrak{h}_{F_\theta^*}^+$  is mapped to  $X_+ + \mathfrak{g}_{F_\theta^*}^+ = X + \mathfrak{g}_{F_\theta^*}^+$ . In other words,  $\overline{X}$  has a representative in  $\mathfrak{h}_{F_\theta^*}$ , so  $(\mathfrak{g}_{F_\theta^*}/\mathfrak{g}_{F_\theta^*}^+)^{\theta-1} \subset \mathfrak{h}_{F_\theta^*}/\mathfrak{h}_{F_\theta^*}^+$ .

Now, let  $\overline{X} \in (\mathfrak{g}_{F_\theta^*}/\mathfrak{g}_{F_\theta^*}^+)^{\theta+1}$ . We have  $\theta(X) + X \in \mathfrak{g}_{F_\theta^*}^+$ , so

$$2X_+ = \theta(X_+ + X_-) + (X_+ + X_-) \in \mathfrak{g}_{F_\theta^*}^+.$$

Thus, since  $p \neq 2$ , we have  $X_+ \in \mathfrak{g}_{F_\theta^*}^+$ . Thus, the coset  $X_- + \mathfrak{p}_{F_\theta^*}^+$  is mapped to  $X + \mathfrak{g}_{F_\theta^*}^+$ . The inclusion  $\mathfrak{p}_{F_\theta^*}^+/\mathfrak{p}_{F_\theta^*}^+ \subset (\mathfrak{g}_{F_\theta^*}/\mathfrak{g}_{F_\theta^*}^+)^{\theta+1}$  is clear. □

**Definition 4.26.** Suppose  $F_\theta^* \in \mathcal{F}_\theta(r)$ , and  $\mathcal{A}$  is an apartment in  $\mathcal{B}(H)$ . Define

$$A(\mathcal{A}, F_\theta^*) := A(F_\theta^* \cap \mathcal{A}, \mathcal{A}).$$

### 4.3 Standard lifts and $r$ -associativity

**Remark 4.27.** Let  $x \in \mathcal{B}(H)$ , and let  $F_\theta^*(x) \in \mathcal{F}_\theta(r)$ . We call a generalized  $r$ -facet  $F^*$  in  $\mathcal{B}(G)$  the standard lift of  $F_\theta^*(x)$  if  $F^*$  is the generalized  $r$ -facet in  $\mathcal{B}(G)$  containing  $x$ , as defined in [[9], 3.2.1] and in Section 4.2.

**Lemma 4.28.** Let  $y \in \mathcal{B}(G)$ . The generalized  $r$ -facet  $F^*(y)$  is  $\theta$ -stable if and only if  $F^*(y) \cap \mathcal{B}(H) \neq \emptyset$ . In particular, if  $F_\theta^*(x) \in \mathcal{F}_\theta(r)$ , then the standard lift  $F^*(x)$  of  $F_\theta^*(x)$  is  $\theta$ -stable.

*Proof.* “ $\Leftarrow$ ”: Let  $z \in F^*(x)$  with  $x \in \mathcal{B}(H)$ . We must verify that  $\theta(z) \in F^*(x)$ . This occurs if and only if

$$(\dagger) \quad \mathfrak{g}_{\theta(z),r} = \mathfrak{g}_{x,r} \text{ and } \mathfrak{g}_{\theta(z),r^+} = \mathfrak{g}_{x,r^+}.$$

Since  $x$  lies in  $\mathcal{B}(H)$ , we have  $\theta(x) = x$ , so  $z \in F^*(\theta(x))$ . Thus, we have

$$\mathfrak{g}_{z,r} = \mathfrak{g}_{\theta(x),r} \text{ and } \mathfrak{g}_{z,r^+} = \mathfrak{g}_{\theta(x),r^+}$$

which, since  $\theta$  is an involution, is equivalent to  $(\dagger)$ .

“ $\Rightarrow$ ”: Let  $F^* = F^*(y)$ , for some  $y \in \mathcal{B}(G)$ . By [[9], Lemma 3.2.11],  $F^*(\delta)$  is a convex, closed,  $\text{stab}_G(F^*)$ -invariant set of  $\mathcal{B}(G)$ . Also, note that  $\theta$  preserves the boundary of  $F^*$ , and  $\theta$  acts on  $\mathcal{B}(G)$  by an isometry. Thus, if  $z \in \overline{F^*} \setminus F^*$ , and  $x \in F^*(\delta)$ , we have

$$\text{dist}(\theta(x), z) = \text{dist}(\theta(x), \theta(z')) = \text{dist}(x, z') \geq \delta,$$



for all  $z' \in \overline{F^*} \setminus F^*$ . In particular,  $F^*(\delta)$  is  $\theta$ -stable. Now, since  $\langle \theta \rangle$  is a finite group of isometries, we apply [[21], 2.3.1] to conclude that  $\langle \theta \rangle$  has a fixed point in  $F^*(\delta)$ , and hence in  $F^*$ .  $\square$

The following proposition gives us a way to translate the work done in Sections 4.1 and 4.2 into the framework of  $\theta$ -stable  $r$ -facets.

**Proposition 4.29.** *Let  $F_{1,\theta}^*, F_{2,\theta}^* \in \mathcal{F}_\theta(r)$ , and let  $\mathcal{A} \subset \mathcal{A}'$  be apartments in  $\mathcal{B}(H)$  and  $\mathcal{B}(G)$ , respectively, such that  $F_{i,\theta}^* \cap \mathcal{A} \neq \emptyset$ , for  $i = 1, 2$ . If  $F_1^*$  and  $F_2^*$  are the standard lifts of  $F_{1,\theta}^*$  and  $F_{2,\theta}^*$ , respectively, then*

$$A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, F_{2,\theta}^*) \Leftrightarrow A(\mathcal{A}', F_1^*) = A(\mathcal{A}', F_2^*).$$

*Proof.* “ $\Leftarrow$ ” : We first claim that  $A(\mathcal{A}', F_1^*)^\theta = A(\mathcal{A}, F_{1,\theta}^*)$ . If this is not true, then, since  $A(\mathcal{A}', F_1^*)^\theta$  is convex, there is an  $(r, \theta)$ -facet  $C$  in  $A(\mathcal{A}', F_1^*)^\theta$  properly containing  $F_{1,\theta}^* \cap \mathcal{A}$  in its closure. Let  $C'$  be the  $r$ -facet in  $\mathcal{A}'$  containing  $C$ . Since  $F_{1,\theta}^* \cap \mathcal{A} \subset \overline{C}$ , we have  $F_1^* \cap \mathcal{A}' \subset \overline{C'}$ . Note that  $\overline{C'}$  is the union of  $C'$  and  $r$ -facets of strictly smaller dimension. Thus, if  $\dim F_1^* = \dim C'$ , then we must have  $F_1^* = C'$ . By our choice of  $C'$ ,  $\dim F_1^* \neq \dim C'$ . In particular,  $C'$  is a  $\theta$ -stable  $r$ -facet (contained in  $A(\mathcal{A}', F_1^*)$  by Remark 4.1) of strictly larger dimension than  $F_{1,\theta}^* \cap \mathcal{A}$ , a contradiction.

“ $\Rightarrow$ ” : Since  $A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, F_{2,\theta}^*)$ , in particular, we know  $F_1^* \cap \mathcal{A}'$  intersects  $A(\mathcal{A}', F_2^*)$  nontrivially. Thus, by Remark 4.1,  $F_1^* \cap \mathcal{A}'$  is entirely contained in  $A(\mathcal{A}', F_2^*)$ . Similarly,  $F_2^* \cap \mathcal{A}'$  is entirely contained inside  $A(\mathcal{A}', F_1^*)$ .  $\square$

**Definition 4.30.** *Let  $F_{1,\theta}^*, F_{2,\theta}^* \in \mathcal{F}_\theta(r)$ . We say that  $F_{1,\theta}^*$  and  $F_{2,\theta}^*$  are strongly  $r$ -associated if for all apartments  $\mathcal{A}$  in  $\mathcal{B}(H)$  such that  $F_{1,\theta}^* \cap \mathcal{A}$  and  $F_{2,\theta}^* \cap \mathcal{A}$  are nonempty, we have*

$$A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, F_{2,\theta}^*).$$

**Remark 4.31.** *By [[9], Lemma 3.3.3] and Proposition 4.29, this is the same as demanding that the standard lifts  $F_1^*, F_2^*$  be strongly  $r$ -associated in the sense of [[9], Definition 3.3.2].*

**Lemma 4.32.** *Two generalized  $(r, \theta)$ -facets  $F_{1,\theta}^*, F_{2,\theta}^* \in \mathcal{F}_\theta(r)$  are strongly  $r$ -associated if and only if there exists an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  for which  $F_{1,\theta}^* \cap \mathcal{A}, F_{2,\theta}^* \cap \mathcal{A}$  are nonempty, and*

$$A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, F_{2,\theta}^*).$$

*Proof.* “ $\Rightarrow$ ” : The definition of strong  $r$ -associativity proves the forward implication.

“ $\Leftarrow$ ” : Let  $\tilde{\mathcal{A}}$  be an apartment for which  $A(\tilde{\mathcal{A}}, F_{1,\theta}^*) = A(\tilde{\mathcal{A}}, F_{2,\theta}^*) \neq \emptyset$ . By [[9], Lemma 3.3.3] and Proposition 4.29, we have that the standard lifts  $F_1^*$  and  $F_2^*$  (of  $F_{1,\theta}^*$  and  $F_{2,\theta}^*$  respectively) are strongly  $r$ -associated. By Proposition 4.29, the result follows by taking  $\theta$ -fixed points.  $\square$

**Definition 4.33.** *Two generalized  $(r, \theta)$ -facets  $F_{1,\theta}^*, F_{2,\theta}^* \in \mathcal{F}_\theta(r)$  are said to be  $r$ -associated if there exists an  $h \in H$  such that  $F_{1,\theta}^*$  and  $hF_{2,\theta}^*$  are strongly  $r$ -associated.*

**Lemma 4.34.**  *$r$ -associativity is an equivalence relation on  $\mathcal{F}_\theta(r)$ . Whenever two generalized  $(r, \theta)$ -facets  $F_{1,\theta}^*, F_{2,\theta}^*$  are  $r$ -associated, we write  $F_{1,\theta}^* \sim F_{2,\theta}^*$ .*

*Proof.* For reflexivity, let  $F_\theta^* \in \mathcal{F}_\theta(r)$ . Suppose  $\mathcal{A} \subset \mathcal{B}(H)$  such that  $\mathcal{A} \cap F_\theta^* \neq \emptyset$ . We have  $A(\mathcal{A}, F_\theta^*) = A(\mathcal{A}, F_\theta^*)$ , so the relation is reflexive.

Now, suppose  $F_{1,\theta}^* \sim F_{2,\theta}^*$ . Then, there exists an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  and an element  $h \in H$  such that

$$A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, hF_{2,\theta}^*) \neq \emptyset.$$

Recall  $hA(\mathcal{A}, F_\theta^*) = A(h\mathcal{A}, hF_\theta^*)$  for any  $h \in H$ . Thus, multiplying the equation above by  $h^{-1}$ , we obtain

$$A(h^{-1}\mathcal{A}, h^{-1}F_{1,\theta}^*) = A(h^{-1}\mathcal{A}, F_{2,\theta}^*) \neq \emptyset.$$

In particular,  $F_{2,\theta}^* \sim F_{1,\theta}^*$ .

Now, suppose  $F_{1,\theta}^*, F_{2,\theta}^*$ , and  $F_{3,\theta}^*$  are generalized  $(r, \theta)$ -facets such that  $F_{1,\theta}^* \sim F_{2,\theta}^*$  and  $F_{2,\theta}^* \sim F_{3,\theta}^*$ . Then, by definition, there exist  $h_2, h_3 \in H$  and apartments  $\mathcal{A}_{12}, \mathcal{A}_{23} \subset \mathcal{B}(H)$  such that

$$A(\mathcal{A}_{12}, F_{1,\theta}^*) = A(\mathcal{A}_{12}, h_2F_{2,\theta}^*) \neq \emptyset$$

and

$$A(\mathcal{A}_{23}, F_{2,\theta}^*) = A(\mathcal{A}_{23}, h_3F_{3,\theta}^*) \neq \emptyset.$$

Let  $z \in C(F_{2,\theta}^*)$ . Then, since  $h_2^{-1}\mathcal{A}_{12} \cap F_{2,\theta}^* \neq \emptyset$  and  $\mathcal{A}_{23} \cap F_{2,\theta}^* \neq \emptyset$ ,  $z$  lies in  $h_2^{-1}\mathcal{A}_{12} \cap \mathcal{A}_{23}$ . By Remark 4.11, there exists some  $h \in H_z \subset \text{stab}_H(F_{2,\theta}^*)$  (since  $h$  fixes  $z$ ) such that  $hh_2^{-1}\mathcal{A}_{12} = \mathcal{A}_{23}$ .

Using these facts, we have

$$\begin{aligned} \emptyset \neq A(\mathcal{A}_{12}, F_{1,\theta}^*) &= A(\mathcal{A}_{12}, h_2F_{2,\theta}^*) = h_2A(h_2^{-1}\mathcal{A}_{12}, F_{2,\theta}^*) \\ &= h_2h^{-1}A(\mathcal{A}_{23}, hF_{2,\theta}^*) = h_2h^{-1}A(\mathcal{A}_{23}, F_{2,\theta}^*) \\ &= A(\mathcal{A}_{12}, h_2h^{-1}h_3F_{3,\theta}^*). \end{aligned}$$

□

## 4.4 Identification of some $\mathfrak{f}$ -vector spaces

**Definition 4.35.** *As in [[9], 3.4.1], for  $x \in \mathcal{B}(G)$ , let  $V_{x,r}$  denote the  $\mathfrak{f}$ -vector space  $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ .*

**Definition 4.36.** *If  $F_\theta^* \in \mathcal{F}_\theta(r)$  and  $x \in F_\theta^*$ , define  $V_{F_\theta^*} := V_{x,r}$ .*

Using Lemma 4.25, we identify  $V_{x,r}^+$  with the image of  $\mathfrak{h}_{x,r}/\mathfrak{h}_{x,r^+}$ . In particular, we interpret the quotient map given by  $\mathfrak{h}_{x,r} \rightarrow V_{x,r}^+$  below using this identification.

**Lemma 4.37.** *Suppose  $F_{1,\theta}^*, F_{2,\theta}^* \in \mathcal{F}_\theta(r)$  are strongly  $r$ -associated. Then the natural maps*

$$\mathfrak{h}_{F_{1,\theta}^*} \cap \mathfrak{h}_{F_{2,\theta}^*} \rightarrow V_{F_{i,\theta}^*}^+$$

$$\mathfrak{p}_{F_{1,\theta}^*} \cap \mathfrak{p}_{F_{2,\theta}^*} \rightarrow V_{F_{i,\theta}^*}^-$$

are surjective with kernels  $\mathfrak{h}_{F_{1,\theta}^*}^+ \cap \mathfrak{h}_{F_{2,\theta}^*}^+ = \mathfrak{h}_{F_{1,\theta}^*}^+ \cap \mathfrak{h}_{F_{2,\theta}^*}^+ = \mathfrak{h}_{F_{1,\theta}^*}^+ \cap \mathfrak{h}_{F_{2,\theta}^*}^+$  and  $\mathfrak{p}_{F_{1,\theta}^*}^+ \cap \mathfrak{p}_{F_{2,\theta}^*}^+ = \mathfrak{p}_{F_{1,\theta}^*}^+ \cap \mathfrak{p}_{F_{2,\theta}^*}^+$ , respectively.

*Proof.* By Remark 4.31, we know that the standard lifts  $F_1^*$  and  $F_2^*$  are strongly  $r$ -associated. Let  $e \in V_{F_{i,\theta}^*}^+$ . By [[9], 3.5.1], there is a lift  $X \in \mathfrak{g}_{F_{1,\theta}^*} \cap \mathfrak{g}_{F_{2,\theta}^*}$ . Let  $X_+$  denote the projection of  $X$  to  $\mathfrak{h}_{F_{i,\theta}^*}^+$ . Then, by the proof of Lemma 4.25,  $X_+$  is mapped to  $e$ . By Lemma 4.7,  $X_+$  lies in  $\mathfrak{h} \cap \mathfrak{g}_{F_{1,\theta}^*} \cap \mathfrak{g}_{F_{2,\theta}^*} = \mathfrak{h}_{F_{1,\theta}^*} \cap \mathfrak{h}_{F_{2,\theta}^*}$ . Thus, the map is surjective. If  $X$  lies in the kernel of the first map, then, by [[9], 3.5.1],  $X$  is contained in  $\mathfrak{g}_{F_{1,\theta}^*}^+ \cap \mathfrak{g}_{F_{2,\theta}^*}^+ = \mathfrak{g}_{F_{1,\theta}^*}^+ \cap \mathfrak{g}_{F_{2,\theta}^*}^+ = \mathfrak{g}_{F_{1,\theta}^*}^+ \cap \mathfrak{g}_{F_{2,\theta}^*}^+$ . Thus, the kernel is  $\mathfrak{h} \cap \mathfrak{g}_{F_{1,\theta}^*}^+ \cap \mathfrak{g}_{F_{2,\theta}^*}^+ = \mathfrak{h}_{F_{1,\theta}^*}^+ \cap \mathfrak{h}_{F_{2,\theta}^*}^+ = \mathfrak{h}_{F_{1,\theta}^*}^+ \cap \mathfrak{h}_{F_{2,\theta}^*}^+$ .

Similarly, let  $X \in \mathfrak{p}_{F_{1,\theta}^*} \cap \mathfrak{p}_{F_{2,\theta}^*}$ , and suppose  $X$  is mapped to the trivial coset in  $V_{F_{i,\theta}^*}^-$ . Then, again by [[9], 3.5.1], we have  $X \in \mathfrak{p}_{F_{1,\theta}^*}^+ \cap \mathfrak{p}_{F_{2,\theta}^*}^+ = \mathfrak{p}_{F_{1,\theta}^*}^+ \cap \mathfrak{p}_{F_{2,\theta}^*}^+ = \mathfrak{p}_{F_{1,\theta}^*}^+ \cap \mathfrak{p}_{F_{2,\theta}^*}^+$ . Let  $e$  be a coset in  $V_{F_{i,\theta}^*}^-$ . By the same result, there exists a lift  $X \in \mathfrak{g}_{F_{1,\theta}^*} \cap \mathfrak{g}_{F_{2,\theta}^*}$ . By Proposition 4.8, we may project  $X$  to  $X_- \in \mathfrak{p}_{F_{i,\theta}^*}$ . This is the desired lift which lies in  $\mathfrak{p}_{F_{1,\theta}^*} \cap \mathfrak{p}_{F_{2,\theta}^*}$ .  $\square$

**Remark 4.38.** Due to the previous result, whenever  $F_{1,\theta}^*$  and  $F_{2,\theta}^*$  are strongly  $r$ -associated, we are able to identify  $V_{F_{1,\theta}^*}^+$  with  $V_{F_{2,\theta}^*}^+$  and  $V_{F_{1,\theta}^*}^-$  with  $V_{F_{2,\theta}^*}^-$ . We let  $i^+$  and  $i^-$  denote the respective bijective identifications.

**Definition 4.39.** If  $F_\theta^* \in \mathcal{F}_\theta(r)$  and  $x \in F_\theta^*$ , then the image of  $H_x$  in  $\text{Aut}_f(V_{F_\theta^*}^-)$  is denoted by  $N_x^-(F_\theta^*)$ .

**Lemma 4.40.** Suppose  $F_{i,\theta}^* \in \mathcal{F}_\theta(r)$  and  $x_i \in F_{i,\theta}^*$  for  $i = 1, 2$ . If  $F_{1,\theta}^*$  and  $F_{2,\theta}^*$  are strongly  $r$ -associated, then  $N_{x_i}^-(F_{i,\theta}^*)$  is the image of  $H_{x_1} \cap H_{x_2}$  in  $\text{Aut}_f(V_{F_{i,\theta}^*}^-)$  for  $i = 1, 2$ . Moreover,

$$N_{x_1}^-(F_{1,\theta}^*) = N_{x_2}^-(F_{2,\theta}^*)$$

under the identification induced by  $i^-$ .

*Proof.* Let  $\mathcal{A}$  be an apartment in  $\mathcal{B}(H)$  containing  $x_1$  and  $x_2$ . Choose  $\psi \in \Psi(\mathcal{A})$  such that the image of  $U_\psi$  in  $\text{Aut}_f(V_{F_{1,\theta}^*}^-)$  is nontrivial and  $\psi(x_1) = 0$ . We will show that  $\psi(x_2) = 0$ .

Suppose  $\psi(x_2) > 0$ . Since the image of  $U_\psi$  in  $\text{Aut}_f(V_{F_{1,\theta}^*}^-)$  is nontrivial, using the identification from Lemma 4.37, there exists some  $h \in U_\psi$  and  $X \in \mathfrak{p}_{x_1,r} \cap \mathfrak{p}_{x_2,r}$  such that

$${}^hX \neq X \pmod{\mathfrak{p}_{x_1,r^+} \cap \mathfrak{p}_{x_2,r^+}}.$$

On the other hand, we have  ${}^hX - X \in \mathfrak{p}_{x_2,r^+}$ , so by Lemma 4.37, we have

$${}^hX - X \in \mathfrak{p}_{x_1,r} \cap \mathfrak{p}_{x_2,r^+} = \mathfrak{p}_{x_1,r^+} \cap \mathfrak{p}_{x_2,r^+}.$$

This is a contradiction, so we must have  $\psi(x_2) \leq 0$ .

Suppose  $\psi(x_2) < 0$ . Since  $x_1$  and  $x_2$  lie in an affine space, we regard  $v = x_2 - x_1$  as a vector. Consider the function

$$f_v : \mathbb{R} \rightarrow \mathbb{R}$$

defined by  $\epsilon \mapsto \psi(x_1 + \epsilon v)$ , where  $x_1 + \epsilon v$  is interpreted as the point  $z \in \mathcal{A}$  for which  $z - x_1 = \epsilon v$ . For all  $\epsilon \in \mathbb{R}$ , we have  $x_1 + \epsilon v \in \mathcal{A}$ , so  $f_v$  is well-defined. Since  $\psi(x_2) < 0$ , we have

$$f_v(1) = \psi(x_2) < 0,$$

so, since  $\psi$  is continuous, we must have  $f_v(\epsilon) < 0$  whenever  $\epsilon > 0$ , and similarly  $f_v(\epsilon) > 0$  whenever  $\epsilon < 0$ . Recall that  $A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, F_{2,\theta}^*)$  by definition of strong  $r$ -associativity. Since  $x_1, x_2 \in A(\mathcal{A}, F_{2,\theta}^*) = A(\mathcal{A}, F_{1,\theta}^*)$ , we must have  $x_1 + \mathbb{R}v \subset A(\mathcal{A}, F_{1,\theta}^*)$ . Thus, since  $F_{1,\theta}^* \cap \mathcal{A}$  is open in  $A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, F_{2,\theta}^*)$ , there is some  $\epsilon < 0$  for which  $x_1 + \epsilon v \in F_{1,\theta}^* \cap \mathcal{A}$ . In particular, by Lemma 4.6, we have  $\mathfrak{p}_{x_1+\epsilon v, r} = \mathfrak{p}_{x_1, r}$  and  $\mathfrak{p}_{x_1+\epsilon v, r^+} = \mathfrak{p}_{x_1, r^+}$ . Moreover,  $\psi(x_1 + \epsilon v) = f_v(\epsilon) > 0$ , so we have  $U_\psi \subset H_{x_1+\epsilon v, 0^+}$ . Thus,  $h \in U_\psi$  acts trivially on  $\mathfrak{p}_{x_1+\epsilon v, r} / \mathfrak{p}_{x_1+\epsilon v, r^+} = V_{F_{1,\theta}^*}^-$ , a contradiction.

We have thus shown that  $\psi(x_2) = 0$ . If  $\mathcal{A}$  corresponds to a maximal  $k$ -split torus  $\mathbf{S}$  of  $\mathbf{H}$ , recall that  $\mathbf{S}$  lies inside a maximal  $k$ -torus  $\mathbf{Z}$  as described in Section 2.3. Since the image of  $H_{x_1}$  is determined by a filtration subgroup of  $\mathbf{Z}$  (which also lies in  $H_{x_2}$ ) and the  $U_\psi$ 's, the proof shows that if  $h \in H_{x_1}$  has nontrivial image in  $\text{Aut}_{\mathfrak{f}}(V_{F_{1,\theta}^*}^-)$ , then there exists some  $h' \in H_{x_1} \cap H_{x_2}$  for which the images of  $h$  and  $h'$  in  $\text{Aut}_{\mathfrak{f}}(V_{F_{1,\theta}^*}^-)$  coincide.  $\square$

**Definition 4.41.** Let  $F_\theta^* \in \mathcal{F}_\theta(r)$  and  $x \in F_\theta^*$ . Define  $N^-(F_\theta^*) \subset \text{Aut}_{\mathfrak{f}}(V_{F_\theta^*}^-)$  by

$$N^-(F_\theta^*) = N_x^-(F_\theta^*).$$

## 4.5 An equivalence relation

**Definition 4.42.**

$$I_r := \{(F_\theta^*, v) \mid F_\theta^* \in \mathcal{F}_\theta(r) \text{ and } v \in V_{F_\theta^*}^-\}$$

Let  $x \in \mathcal{B}(H)$ . For  $v \in V_{x,r}$ , we interpret  ${}^h v$  as the image of  ${}^h X$ , where  $X$  is a lift of  $v$  in  $\mathfrak{g}_{x,r}$ .

**Definition 4.43.** For  $(F_{1,\theta}^*, v_1)$  and  $(F_{2,\theta}^*, v_2)$  in  $I_r$ , we write  $(F_{1,\theta}^*, v_1) \sim (F_{2,\theta}^*, v_2)$  provided that there exists some  $h \in H$  and an apartment  $\mathcal{A} \subset \mathcal{B}(H)$ , for which  $F_{1,\theta}^* \cap \mathcal{A}, F_{2,\theta}^* \cap \mathcal{A}$  are nonempty, and

1.  $A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, hF_{2,\theta}^*)$  and
2.  $v_1 = {}^h v_2$  in  $V_{F_{1,\theta}^*}^- = V_{hF_{2,\theta}^*}^-$ ,

where we use the usual identification from Lemma 4.37 for the second condition.

**Lemma 4.44.** The relation defined above is an equivalence relation on  $I_r$ .

*Proof.* For reflexivity, let  $h = 1$ .

Now, suppose  $(F_{1,\theta}^*, v_1) \sim (F_{2,\theta}^*, v_2)$ . By definition, there exists an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  and an element  $h \in H$  such that

1.  $A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, hF_{2,\theta}^*) \neq \emptyset$  and
2.  $v_1 = {}^h v_2$  in  $V_{F_{1,\theta}^*}^- = V_{hF_{2,\theta}^*}^-$ .

Since  $h^{-1}A(\mathcal{A}, F_{1,\theta}^*) = A(h^{-1}\mathcal{A}, h^{-1}F_{2,\theta}^*)$ , we have

1.  $A(h^{-1}\mathcal{A}, h^{-1}F_{1,\theta}^*) = A(h^{-1}\mathcal{A}, F_{2,\theta}^*) \neq \emptyset$  and
2.  ${}^{h^{-1}}v_1 = v_2$  in  $V_{h^{-1}F_{1,\theta}^*}^- = V_{F_{2,\theta}^*}^-$ .

In particular,  $(F_{2,\theta}^*, v_2) \sim (F_{1,\theta}^*, v_1)$ .

For transitivity, suppose  $(F_{1,\theta}^*, v_1), (F_{2,\theta}^*, v_2), (F_{3,\theta}^*, v_3) \in I_r$  such that  $(F_{1,\theta}^*, v_1) \sim (F_{2,\theta}^*, v_2)$  and  $(F_{2,\theta}^*, v_2) \sim (F_{3,\theta}^*, v_3)$ . By definition, there exist  $h_2, h_3 \in H$  and apartments  $\mathcal{A}_{12}, \mathcal{A}_{23} \subset \mathcal{B}(H)$  such that

$$\begin{aligned} A(\mathcal{A}_{12}, F_{1,\theta}^*) &= A(\mathcal{A}_{12}, h_2 F_{2,\theta}^*) \neq \emptyset \\ A(\mathcal{A}_{23}, F_{2,\theta}^*) &= A(\mathcal{A}_{23}, h_3 F_{3,\theta}^*) \neq \emptyset \end{aligned}$$

and

$$\begin{aligned} v_1 &= {}^{h_2} v_2 \text{ in } V_{F_{1,\theta}^*}^- = V_{h_2 F_{2,\theta}^*}^- \\ v_2 &= {}^{h_3} v_3 \text{ in } V_{F_{2,\theta}^*}^- = V_{h_3 F_{3,\theta}^*}^- \end{aligned}$$

Fix  $x_i \in C(F_{i,\theta}^*)$ . The first and second lines show that  $A(\mathcal{A}_{12}, h_2 F_{2,\theta}^*), A(h_2 \mathcal{A}_{23}, h_2 F_{2,\theta}^*) \neq \emptyset$ . In particular,  $\mathcal{A}_{12} \cap h_2 F_{2,\theta}^*, h_2 \mathcal{A}_{23} \cap h_2 F_{2,\theta}^* \neq \emptyset$ , so there exists an element  $h \in H_{h_2 x_2} \subset \text{stab}_H(h_2 F_{2,\theta}^*)$  such that  $h \mathcal{A}_{12} = h_2 \mathcal{A}_{23}$ . We have

$$\begin{aligned} \emptyset \neq A(\mathcal{A}_{12}, F_{1,\theta}^*) &= A(\mathcal{A}_{12}, h_2 F_{2,\theta}^*) = h^{-1} A(h \mathcal{A}_{12}, h h_2 F_{2,\theta}^*) \\ &= h^{-1} A(h_2 \mathcal{A}_{23}, h_2 F_{2,\theta}^*) = h^{-1} h_2 A(\mathcal{A}_{23}, h_3 F_{3,\theta}^*) \\ &= A(\mathcal{A}_{12}, h^{-1} h_2 h_3 F_{3,\theta}^*). \end{aligned}$$

Now, by the proof of [[9], 3.5.1] and Lemma 4.37, we have a surjection

$$\mathfrak{p}_{F_{1,\theta}^*} \cap \mathfrak{p}_{h_2 F_{2,\theta}^*} \cap \mathfrak{p}_{h^{-1} h_2 h_3 F_{3,\theta}^*} \rightarrow V_{F_{1,\theta}^*}^-.$$

As a result, there is some  $X \in \mathfrak{p}_{F_{1,\theta}^*} \cap \mathfrak{p}_{h_2 F_{2,\theta}^*} \cap \mathfrak{p}_{h^{-1} h_2 h_3 F_{3,\theta}^*}$  such that the image of  $X$  in  $V_{F_{1,\theta}^*}^-$  is  $v_1$ . Since  ${}^{h_2} v_2 = v_1$  under the standard identification, we have that the image of  $X$  in  $V_{h_2 F_{2,\theta}^*}^-$  is  ${}^{h_2} v_2$ , so the image of  ${}^{h_2^{-1}} X$  in  $V_{F_{2,\theta}^*}^-$  is  $v_2$ . Recall that  $h \in H_{h_2 x_2} = \text{Int}(h_2) H_{x_2}$ , so  $h_2^{-1} h h_2 \in H_{x_2}$ . Thus, the image of  ${}^{h_2^{-1}} h X = ({}^{h_2^{-1}} h h_2) {}^{h_2^{-1}} X$  in  $V_{F_{2,\theta}^*}^-$  is  ${}^{h_2^{-1}} h h_2 v_2$ . By the previous computation and the fact that  $X$  was chosen in  $\mathfrak{p}_{h^{-1} h_2 h_3 F_{3,\theta}^*}$ , we have  ${}^{h_2^{-1}} h X \in \mathfrak{p}_{F_{2,\theta}^*} \cap \mathfrak{p}_{h_3 F_{3,\theta}^*}$ .

Since  $F_{2,\theta}^*$  and  $h_3 F_{3,\theta}^*$  are strongly  $r$ -associated, by Lemma 4.40,  $N^-(F_{2,\theta}^*) = N^-(h_3 F_{3,\theta}^*)$ . Thus, again by Lemma 4.40, there is an element  $h' \in H_{h_3 x_3} \cap H_{x_2}$  such that

$$h_2^{-1} h h_2 v_2 = h' v_2 = h' h_3 v_3 \text{ in } V_{F_{2,\theta}^*}^- = V_{h_3 F_{3,\theta}^*}^-.$$

As a consequence, the image of  $X$  in  $V_{h^{-1} h_2 h_3 F_{3,\theta}^*}^-$  is

$$h^{-1} h_2 h' h_3 v_3 = h^{-1} (h_2 h' h_2^{-1}) h_2 h_3 v_3 = h'' h^{-1} h_2 h_3 v_3$$

with  $h'' \in \text{Int}(h^{-1} h_2)(H_{h_3 x_3} \cap H_{x_2}) \subset H_{h^{-1} h_2 h_3 x_3}$ . Since  $h'' \in H_{h^{-1} h_2 h_3 x_3} \subset \text{stab}_H(h^{-1} h_2 h_3 F_{3,\theta}^*)$ , we have

$$A(\mathcal{A}_{12}, F_{1,\theta}^*) = A(\mathcal{A}_{12}, h^{-1} h_2 h_3 F_{3,\theta}^*) = A(\mathcal{A}_{12}, h'' h^{-1} h_2 h_3 F_{3,\theta}^*) \neq \emptyset.$$

Moreover,

$$v_1 = h_2 v_2 = h'' h^{-1} h_2 h_3 v_3 \text{ in } V_{F_{1,\theta}^*}^- = V_{h'' h^{-1} h_2 h_3 F_{3,\theta}^*}^-.$$

This shows that  $(F_{1,\theta}^*, v_1) \sim (F_{3,\theta}^*, v_3)$ . □

## 5 Jacobson-Morosov triples over $\mathfrak{f}$ and $k$

Fix  $r \in \mathbb{R}$ . Before attaching a nilpotent  $H$ -orbit to the types of pairs discussed at the end of Section 4, we will need a way to pass from  $\mathfrak{sl}_2(\mathfrak{f})$ -triples to  $\mathfrak{sl}_2(k)$ -triples and vice versa. In this section, we describe this procedure in detail. Recall that  $\mathcal{N}$  denotes the set of nilpotent elements in  $\mathfrak{g}$  as defined in the preliminaries. As in Lemma 4.25, we will identify  $\mathfrak{p}_{x,r}/\mathfrak{p}_{x,r+}$  with  $V_{x,r}^-$ .

**Definition 5.1.** *Let  $F_\theta^* \in \mathcal{F}_\theta(r)$ . An element  $e \in V_{F_\theta^*}^-$  is called degenerate provided that there exists a lift  $E \in \mathfrak{p}_{F_\theta^*} \cap \mathcal{N}$ .*

The following lemma gives us an alternate characterization of degenerate elements.

**Lemma 5.2.** *Fix  $F_\theta^* \in \mathcal{F}_\theta(r)$ . An element  $e \in V_{F_\theta^*}^-$  is degenerate if and only if zero lies in the Zariski closure of  $H_x e$  for all  $x \in F_\theta^*$ .*

*Proof.* “ $\Rightarrow$ ”: We refer to [[17], Proposition 4.3]. Fix  $x \in F_\theta^*$  and a lift  $E \in \mathfrak{p}_{x,r} \cap \mathcal{N}$ . In the notation of [[17], Proposition 4.3], we take  $V = \mathfrak{p}_{x,r}$ ,  $W = \mathfrak{p}_{x,r+}$ , and let  $\rho : H_x \rightarrow GL(V)$  be the rational representation given by the adjoint action of  $H_x$  on  $\mathfrak{p}$  restricted to the lattice  $\mathfrak{p}_{x,r}$ . We note that  $\varpi \mathfrak{p}_{x,r} = \mathfrak{p}_{x,r+1} \subset \mathfrak{p}_{x,r+}$ , and  $E$  is nilpotent lift of  $e$ , so all hypotheses are satisfied. From [[17], Proposition 4.3], we conclude that zero lies in the Zariski closure of  $H_x e$ , with respect to the induced representation of  $\rho$  from  $H_x$  to  $GL(V/W)$ .

“ $\Leftarrow$ ”: Fix  $x \in F_\theta^*$ . Let  $\mathbf{S}$  be a maximal  $k$ -split torus in  $\mathbf{H}$  with  $x \in \mathcal{A}(\mathbf{S}, k)$ . We consider  $V_{F_\theta^*}^-$  as the vector space of  $\mathfrak{f}$ -rational points of the affine  $H_x$ -scheme  $\text{Lie}(\mathbf{G}_x)^-$ . Then, by [[13], Theorem 1.4], there exists a one-parameter subgroup  $\bar{\lambda} \in \mathbf{X}_*^{\mathfrak{f}}(H_x)$  such that

$$\lim_{t \rightarrow 0} \bar{\lambda}(t) e = 0.$$

Let  $\mathbf{S}$  be the maximal  $\mathfrak{f}$ -split torus in  $H_x$  corresponding to  $\mathbf{S}$ . Then, since  $H_x(\mathfrak{f})$  acts transitively on the set of maximal  $\mathfrak{f}$ -split tori in  $H_x$ , there exists an element  $\bar{h} \in H_x(\mathfrak{f})$  and a one-parameter subgroup  $\bar{\mu} \in \mathbf{X}_*^f(\mathbf{S})$  such that

$$\lim_{t \rightarrow 0} \bar{\mu}(t) \bar{h} e = 0.$$

Let  $\mu \in \mathbf{X}_*(\mathbf{S})$  be a lift of  $\bar{\mu}$  and let  $h \in H_x$  be a lift of  $\bar{h}$ . Also, let  $E'$  be a lift of  $e$  in  $\mathfrak{p}_{x,r}$ . Without loss of generality, we may assume that

$${}^h E' = \sum_{\psi} X_{\psi}$$

where  $X_{\psi} \in \mathfrak{g}_{\psi} \cap \mathfrak{p}$  and  $\psi(x) = r$ . We claim that  $\psi(x + \epsilon \cdot \mu) > r$  for all  $\psi$  appearing in the sum. This will happen precisely when  $\langle \mu, \dot{\psi} \rangle > 0$  since

$$\psi(x + \epsilon \mu) = \psi(x) + \epsilon \cdot \langle \mu, \dot{\psi} \rangle = r + \epsilon \cdot \langle \mu, \dot{\psi} \rangle.$$

Note, however, that

$$0 = \lim_{t \rightarrow 0} \bar{\mu}(t) \bar{h} E' = \sum \lim_{t \rightarrow 0} \bar{\mu}(t) \overline{X_{\psi}} = \sum \lim_{t \rightarrow 0} t^{\langle \mu, \dot{\psi} \rangle} \overline{X_{\psi}}$$

so, in particular, the limit is 0 if and only if  $\langle \mu, \dot{\psi} \rangle > 0$ . This shows that  ${}^h E' \in \mathfrak{p}_{x+\epsilon\mu, r+}$ . For  $\epsilon$  sufficiently small,  $x + \epsilon\mu$  lies in a generalized  $(r, \theta)$ -facet  $C_{\theta}^*$  containing  $F_{\theta}^*$  in its closure. By [[9], Corollary 3.2.19], we have  $\mathfrak{p}_{x,r+} \subset \mathfrak{p}_{x+\epsilon\mu, r+}$ . Thus,

$${}^h(E' + \mathfrak{p}_{x,r+}) = {}^h(E' + \mathfrak{p}_{F_{\theta}^*}^+) \subset \mathfrak{p}_{x+\epsilon\mu, r+}$$

for  $\epsilon$  taken to be sufficiently small. We have shown that the coset  $E' + \mathfrak{p}_{x,r}$  lies in  $\mathfrak{g}_{r+}$  as defined in [[2], 3.2.5]. In particular, by [[2], Corollary 3.2.6],  $e$  is a degenerate coset.  $\square$

In order to discuss  $\mathfrak{sl}_2(\mathfrak{f})$ -triples, we next introduce an  $\mathfrak{f}$ -Lie algebra  $\bar{\mathfrak{g}}_x$  which is associated to a point  $x \in \mathcal{B}(H)$ . In the preliminaries, we chose a uniformizer  $\varpi$  for  $k$ , which allows us to identify  $V_{x,s}$  with  $V_{x,s+j \cdot \ell}$  where  $L$  is the splitting field of  $\mathbf{G}$  containing  $K$ , and  $\ell = [L : K]$ . Using this identification, we define

$$\bar{\mathfrak{g}}_x := \bigoplus_{s \in \mathbb{R}/\ell \cdot \mathbb{Z}} V_{x,s}.$$

If  $\bar{X}_s \in V_{x,s}$  and  $\bar{X}_t \in V_{x,t}$ , then define  $[\bar{X}_s, \bar{X}_t]$  to be the image of  $[X_s, X_t] \in \mathfrak{g}_{x,(s+t)}$  in  $V_{x,(s+t)}$  where  $X_s \in \mathfrak{g}_{x,s}$  and  $X_t \in \mathfrak{g}_{x,t}$  are any lifts of  $\bar{X}_s$  and  $\bar{X}_t$  respectively. We can then linearly extend to obtain a well-defined bracket on all of  $\bar{\mathfrak{g}}_x$ . With this product,  $\bar{\mathfrak{g}}_x$  is an  $\mathfrak{f}$ -Lie algebra.

## 5.1 Some hypotheses

We now list some hypotheses (which occur also in [[9]]) needed in order to utilize the theory of  $\mathfrak{sl}_2$ -triples and pass from the Lie algebra setting to the group setting when necessary. These hypotheses hold under mild restrictions on  $\mathbf{G}, \mathbf{H}$  and  $k$ , and we give some references

for more details on when each hypothesis is valid. It should be noted that in characteristic 0, all hypotheses hold.

**Hypothesis 5.3.** *Suppose  $x \in \mathcal{B}(H)$ . If  $X \in \mathcal{N} \cap (\mathfrak{p}_{x,r} \setminus \mathfrak{p}_{x,r}^+)$ , then there exist  $H \in \mathfrak{h}_{x,0}$  and  $Y \in \mathfrak{p}_{x,-r}$  such that*

$$\begin{aligned} [H, X] &= 2X \bmod \mathfrak{p}_{x,r}^+ \\ [H, Y] &= -2Y \bmod \mathfrak{p}_{x,(-r)}^+ \\ [X, Y] &= H \bmod \mathfrak{h}_{x,0}^+. \end{aligned}$$

If  $(f, h, e)$  denotes the image of  $(Y, H, X)$  in  $V_{x,-r} \times V_{x,0} \times V_{x,r} \subset \bar{\mathfrak{g}}_x$ , then  $\{f, h, e\}$  is an  $\mathfrak{sl}_2(\mathfrak{f})$ -triple, and  $\bar{\mathfrak{g}}_x$  decomposes into a direct sum of irreducible  $\langle f, h, e \rangle$ -modules of highest weight at most  $p - 3$ . Moreover, there exists some  $\bar{\lambda} \in \mathbf{X}_*^f(H_x)$ , uniquely determined up to an element of  $\mathbf{X}_*(Z_x)$  whose differential is zero, such that the following hold:

1. The image of  $d\bar{\lambda}$  in  $\text{Lie}(H_x)$  coincides with the subspace spanned by  $h$ .
2. Suppose  $i \in \mathbb{Z}$ . For  $v \in \bar{\mathfrak{g}}_x$

$$\text{if } \bar{\lambda}^{(t)}v = t^i v, \text{ then } |i| \leq p - 3 \text{ and } \text{ad}(h)v = iv.$$

More details on Hypothesis 5.3 can be found in [[9], Appendix A].

Following [[16], I.2], we define a *normal*  $\mathfrak{sl}_2$ -triple below.

**Definition 5.4.** *Let  $\{Y, H, X\}$  (resp.  $\{f, h, e\}$ ) be an  $\mathfrak{sl}_2(k)$ -triple in  $\mathfrak{g}$  (resp.  $\mathfrak{sl}_2(\mathfrak{f})$ -triple in  $\bar{\mathfrak{g}}_x$ ). We call  $\{Y, H, X\}$  (resp.  $\{f, h, e\}$ ) a *normal*  $\mathfrak{sl}_2(k)$ -triple (resp.  $\mathfrak{sl}_2(\mathfrak{f})$ -triple) provided that  $X, Y \in \mathfrak{p}$  (resp.  $e, f \in (\bar{\mathfrak{g}}_x)^-$ ) and  $H \in \mathfrak{h}$  (resp.  $h \in (\bar{\mathfrak{g}}_x)^+$ ).*

**Remark 5.5.** *We note that if  $\{f, h, e\}$  is any  $\mathfrak{sl}_2(\mathfrak{f})$ -triple in  $\bar{\mathfrak{g}}_x$  with  $e \in V_{x,r}^-$ , then it is normal. By projecting  $h$  to  $V_{x,0}$ , we may assume  $h \in V_{x,0}$ . By Lemma 4.25, we may write  $h = h^+ + h^-$  where  $h^+ \in V_{x,0}^+$  and  $h^- \in V_{x,0}^-$ . We have  $[h, e] = 2e \in V_{x,r}^-$ , so  $[h, e] = [h^+, e] + [h^-, e] = 2e$ . By Lemma 4.25,  $V_{x,r} = V_{x,r}^+ \oplus V_{x,r}^-$  is direct, so, we have  $[h^-, e] = 0$  since  $[V_{x,0}^-, V_{x,r}^-] \subset V_{x,r}^+$ . By a similar argument, we have  $f \in V_{x,-r}^-$ , so  $\{f, h, e\}$  is a normal triple. This also shows that  $h$  is  $\theta$ -fixed. In particular, the one-parameter subgroup  $\bar{\lambda} \in \mathbf{X}_*^f(G_x)$  has image inside  $H_x$ . Moreover, it is clear that conditions 1) and 2) hold in this context as a consequence of Appendix A in [[9]].*

**Definition 5.6.** *Keeping the above notation, we say that  $\bar{\lambda} \in \mathbf{X}_*^f(H_x)$  is adapted to the  $\mathfrak{sl}_2(\mathfrak{f})$ -triple obtained from the image of  $(Y, H, X)$  in  $V_{x,-r} \times V_{x,0} \times V_{x,r}$ .*

**Hypothesis 5.7.** *If  $X \in \mathcal{N}^-$ , then there exists some  $m \in \mathbb{N}$  with  $m \leq p - 2$  such that  $\text{ad}(X)^m = 0$ .*

**Hypothesis 5.8.** *Choose  $m \in \mathbb{N}$  such that Hypothesis 5.7 holds. Suppose the characteristic of  $k$  is zero or greater than  $m$ . Then there exists a  $G$ -equivariant map  $\exp: \mathcal{N} \rightarrow \mathcal{U}$  such that for all  $X \in \mathcal{N}$ , the adjoint action of  $\exp(X)$  on  $\mathfrak{g}$  is given by:*



$$\mathrm{Ad}(\exp(X)) = \sum_{i=0}^m \frac{(\mathrm{ad}(X))^i}{i!}.$$

In the next hypothesis, we use the letter  $H$  in two different contexts. In the first occurrence, it appears as an element of  $\mathfrak{g}$  which is part of an  $\mathfrak{sl}_2(k)$ -triple. In the last line of the hypothesis, it occurs as the group of  $k$ -rational points of  $\mathbf{H}$ . This notation is unfortunate, but in most cases, the meaning of this symbol will be clear from context.

**Hypothesis 5.9.** *Suppose Hypothesis 5.8 holds. Let  $X \in \mathcal{N}^-$ . There exists a normal  $\mathfrak{sl}_2(k)$ -triple completing  $X$ . Moreover, if  $\{Y, H, X\}$  is a normal  $\mathfrak{sl}_2(k)$ -triple completing  $X$ , then there is an algebraic group homomorphism  $\varphi : \mathbf{SL}_2 \rightarrow \mathbf{G}$  defined over  $k$  such that  $d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$*

$$X, d\varphi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = Y, d\varphi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H, \text{ and for all } t \in k,$$

$$1. \varphi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp(tX) \text{ and}$$

$$2. \varphi \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp(tY).$$

Lastly, (see below), any two normal  $\mathfrak{sl}_2(k)$ -triples which complete  $X$  are conjugate by an element of  $C_H(X)$ .

**Proposition 5.10.** *Assume Hypotheses 5.3 and 5.8 hold. If  $\{Y', H', X\}$  and  $\{Y, H, X\}$  are two normal  $\mathfrak{sl}_2(k)$ -triples completing  $X$ , then there exists an element  $h \in C_H(X)$  for which  $Y' = {}^h Y$  and  $H' = {}^h H$ .*

*Proof.* (a generalization of an argument of Kostant) In order to verify the claim, we slightly modify the notation and argument given in [[15], Theorem 3.6] and [[8], Lemma 3.4.7].

Define  $\mathfrak{h}_X := [\mathfrak{p}, X] \cap C_{\mathfrak{h}}(X)$ , and let  $U, V \in \mathfrak{h}_X$ . Since  $V = [W, X]$ , for some  $W \in \mathfrak{p}$ , and since  $U$  centralizes  $X$ , we have

$$[U, V] = [U, [W, X]] = [X, [W, U]].$$

Since  $W \in \mathfrak{p}$  and  $U \in \mathfrak{h}$ , it follows that  $[W, U] \in \mathfrak{p}$ , so  $[U, V] \in [\mathfrak{p}, X]$ . This shows  $\mathfrak{h}_X$  is a Lie subalgebra of  $\mathfrak{g}$ .

Kostant also shows that every element of  $\mathfrak{g}_X := [\mathfrak{g}, X] \cap C_{\mathfrak{g}}(X)$  is nilpotent. It follows that every element of  $\mathfrak{h}_X$  is nilpotent. (Note that  $\mathfrak{h}_X$  is also invariant under  $\mathrm{ad}(H)$ .) By Hypothesis 5.8, the adjoint action of  $\exp(W)$  for  $W \in \mathfrak{h}_X$  on an element of  $\mathfrak{g}$  is given by  $\mathrm{Ad}(\exp(W)) = \sum_i \frac{(\mathrm{ad}(W))^i}{i!}$ . In particular, we have

$$\mathrm{Ad}(\exp(W))(H) = \sum_i \frac{(\mathrm{ad}(W))^i(H)}{i!} \in H + \mathfrak{h}_X.$$

We will show that for every  $V \in \mathfrak{h}_X$ , there exists some  $W \in \mathfrak{h}_X$  such that  $\mathrm{Ad}(\exp(W))(H) = H + V$ . Define  $C_{\mathfrak{h}}(X)(i) := \{Z \in C_{\mathfrak{h}}(X) \mid [H, Z] = iZ\}$ . Kostant shows that  $\mathfrak{g}_X \subset$

$\oplus_{i=1}^m C_{\mathfrak{g}}(X)(i)$  for some natural number  $m$ . In particular,  $\mathfrak{h}_X \subset \oplus_{i=1}^m C_{\mathfrak{h}}(X)(i)$ . We now construct the element  $W$  inductively.

Set  $W_1 = -V_1$ , where  $V_1$  is the component of  $V$  lying in  $C_{\mathfrak{h}}(X)(1)$ . Then,  $W_1$  lies in  $\mathfrak{h}_X$ , and we have  $\text{ad}(W_1)(H) = -[H, W_1] = -W_1 = V_1$ . Again, using Hypothesis 5.8, we have

$$\begin{aligned} \text{Ad}(\exp(W_1))(H) - (H + V) &= \sum_{i=0}^m \frac{(\text{ad}(W_1))^i(H)}{i!} - (H + V) \\ &= (V_1 - V) + \sum_{i=2}^m \frac{(\text{ad}(W_1))^i(H)}{i!} \in \bigoplus_{i \geq 2} C_{\mathfrak{h}}(X)(i). \end{aligned}$$

The last line results from the fact that the restriction of  $\text{ad}(H)$  to  $\mathfrak{h}_X$  takes strictly positive integral values as eigenvalues.

Thus, we have verified the base case. We now assume that we have constructed elements  $W_j$  such that

1.  $W_j \in \bigoplus_{1 \leq i \leq j} C_{\mathfrak{h}}(X)(i)$
2.  $\text{Ad}(\exp(W_j))(H) - (H + V) \in \bigoplus_{j+1 \leq i \leq m} C_{\mathfrak{h}}(X)(i)$ .

Now, let  $W'_{j+1}$  be the component of  $\text{Ad}(\exp(W_j))(H) - (H + V)$  which lies in  $C_{\mathfrak{h}}(X)(j+1)$ .

Letting  $W_{j+1} = W_j + \frac{1}{j+1} W'_{j+1}$ , it is clear that  $W_{j+1} \in \bigoplus_{1 \leq i \leq j+1} C_{\mathfrak{h}}(X)(i)$ . Moreover, we have

$$\begin{aligned} \text{Ad}(\exp(W_{j+1}))(H) - (H + V) &= \sum_{i=0}^m \frac{(\text{ad}(W_{j+1}))^i(H)}{i!} - (H + V) \\ &= H + [W_{j+1}, H] + \cdots - (H + V) \\ &= H + [W_j, H] + \frac{1}{j+1} [W'_{j+1}, H] + \cdots - (H + V) \\ &= H + [W_j, H] - W'_{j+1} + \cdots - (H + V) \end{aligned}$$

Only the terms with indices up to  $i = 1$  have been expanded in the last line written. If we expand higher terms, we obtain a sum of the form

$$H + [W_j, H] - W'_{j+1} + \frac{[W_j, [W_j, H]]}{2!} + \frac{[\frac{1}{j+1} W'_{j+1}, [W_j, H]]}{2!} - \frac{[W_{j+1}, W'_{j+1}]}{2!} \cdots - (H + V)$$

so it becomes clear that expanding will further will give us the sum including  $\text{Ad}(\exp(W_j))(H) - (H + V)$ ,  $-W'_{j+1}$ , and terms which lie in weight spaces of  $C_{\mathfrak{h}}(X)$  with weights greater than or equal to  $(j+2)$ . Thus, by definition of  $W'_{j+1}$ , we have

$$\text{Ad}(\exp(W_{j+1}))(H) - (H + V) \in \bigoplus_{j+2 \leq i \leq m} C_{\mathfrak{h}}(X)(i).$$

Finally, letting  $W = W_m$ , we have  $\text{Ad}(\exp(W))(H) = H + V$ .

Now, since  $[H', X] = 2X = [H, X]$ , we have  $H' - H \in C_{\mathfrak{h}}(X)$ . On the other hand, since  $[X, Y' - Y] = H' - H$ , we have  $H' - H \in [\mathfrak{p}, X]$ . In particular, we have  $H' - H \in \mathfrak{h}_X$ . By the argument above, there is some  $W \in \mathfrak{h}_X$  such that

$$\mathrm{Ad}(\exp(W))(H) = H + (H' - H) = H'.$$

By the construction of the element  $W$  in the proof, it is clear that  $W$  lies in  $\mathfrak{h}$ , so since  $\exp$  takes  $\mathfrak{h} \cap \mathcal{N}$  into  $H$ , we set  $h = \exp(W)$ . □

**Hypothesis 5.11.** *Let  $x \in \mathcal{B}(H)$ . For all  $s \in \mathbb{R}_{>0}$  and for all  $t \in \mathbb{R}$ , there exists a map  $\phi_x : \mathfrak{h}_{x,s} \rightarrow H_{x,s}$  such that for  $V \in \mathfrak{h}_{x,s}$  and  $W \in \mathfrak{p}_{x,t}$  we have*

$$\phi_x(V)W = W + [V, W] \bmod \mathfrak{p}_{x,(s+t)^+}.$$

Hypothesis 5.11 as stated above is weaker than its counterpart in the group case. More precisely, as in [[1], 1.3-1.7], suppose  $x \in \mathcal{B}(G)$ . Then for all  $s \in \mathbb{R}_{>0}$  and for all  $t \in \mathbb{R}$  there exists a map  $\phi_x : \mathfrak{g}_{x,s} \rightarrow G_{x,s}$  such that for  $V \in \mathfrak{g}_{x,s}$  and  $W \in \mathfrak{g}_{x,t}$  we have

$$\phi_x(V)W = W + [V, W] \bmod \mathfrak{g}_{x,(s+t)^+}.$$

From the above equation, we can derive Hypothesis 5.11 provided that the restriction of  $\phi_x$  to  $\mathfrak{h}_{x,s}$  maps into  $H_{x,s}$ . For more details on this assumption, see [[11], Appendix B].

## 5.2 Obtaining $\mathfrak{sl}_2(k)$ -triples from $\mathfrak{sl}_2(\mathfrak{f})$ -triples

Our next step will be to show how to obtain a normal  $\mathfrak{sl}_2(k)$ -triple from a normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple. We first recall the setup.

Let  $x \in \mathcal{B}(H)$  and suppose  $(f, h, e) \subset V_{x,-r}^- \times V_{x,0}^+ \times V_{x,r}^- \subset \bar{\mathfrak{g}}_x$  is a nontrivial normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple. Suppose  $\bar{\mu} \in \mathbf{X}_*^{\mathfrak{f}}(H_x)$  is adapted to  $\{f, h, e\}$ . Let  $\mathbf{S}$  be a maximal  $k$ -split torus of  $\mathbf{H}$  such that  $x \in \mathcal{A}(\mathbf{S}, k)$ . Let  $\mathbf{S}$  be the maximal  $\mathfrak{f}$ -split torus in  $\mathbf{G}_x$  corresponding to  $\mathbf{S}$ . Since  $H_x$  is a reductive group over  $\mathfrak{f}$ , all maximal  $\mathfrak{f}$ -split tori are  $H_x(\mathfrak{f})$ -conjugate, so, in particular, there is a one-parameter subgroup  $\bar{\lambda} \in \mathbf{X}_*(\mathbf{S})$  and an element  $\bar{h} \in H_x(\mathfrak{f})$  with  $\bar{\lambda} = \bar{h}\bar{\mu}$ . Now, let  $\lambda \in \mathbf{X}_*(\mathbf{S})$  be a lift of  $\bar{\lambda}$  and substitute  $\{\bar{h}f, \bar{h}h, \bar{h}e\}$  for  $\{f, h, e\}$ . Under the action of  $\lambda$  we have the following grading on the Lie algebra  $\mathfrak{g}$ :

$$\mathfrak{g}(i) := \{X \in \mathfrak{g} \mid \lambda^{(t)}X = t^i \cdot X\} \text{ and } \bar{\mathfrak{g}}_x(i) := \{v \in \bar{\mathfrak{g}}_x \mid \bar{\lambda}^{(t)}v = t^i \cdot v\}.$$

For  $s \in \mathbb{R}$ , we have analogous gradings on  $\mathfrak{g}_{x,s}$  and  $V_{x,s}$  defined by

$$\mathfrak{g}_{x,s}(i) := \{Z \in \mathfrak{g}_{x,s} \mid \lambda^{(t)}Z = t^i \cdot Z\} \text{ and } V_{x,s}(i) := \{v \in V_{x,s} \mid \bar{\lambda}^{(t)}v = t^i \cdot v\}.$$

Define  $\mathfrak{p}(i) := \mathfrak{p} \cap \mathfrak{g}(i)$ ,  $\mathfrak{p}_{x,r}(i) := \mathfrak{p}_{x,r} \cap \mathfrak{g}(i)$ , and  $V_{x,r}^-(i) := V_{x,r}^- \cap V_{x,r}(i)$ .

**Remark 5.12.** *We recall that the Lie bracket on  $\mathfrak{g}$  does not preserve  $\mathfrak{p}$ . In fact, we have  $[V, W] \in \mathfrak{h}$  for all  $V, W \in \mathfrak{p}$ . In particular, if  $X \in \mathfrak{p}$ , and  $Y \in \mathfrak{p}$ , the element  $\mathrm{ad}(X)^2(Y)$  lies in  $\mathfrak{p}$ . This shows why the map in the following lemma is well-defined.*

**Lemma 5.13.** *Suppose Hypothesis 5.3 holds. If  $X \in \mathfrak{p}_{x,r}(2)$  is a lift of  $e$ , then, for all  $s \in \mathbb{R}$ , the map*

$$\mathrm{ad}(X)^2 : \mathfrak{p}_{x,s-r}(-2) \rightarrow \mathfrak{p}_{x,s+r}(2)$$

*is an isomorphism of  $R$ -modules.*

*Proof.* By [[9], Lemma 4.3.1], we know that the map  $\text{ad}(X)^2 : \mathfrak{g}_{x,s-r}(-2) \rightarrow \mathfrak{g}_{x,s+r}(2)$  is an isomorphism of  $R$ -modules. Thus,

$$\text{ad}(X)^2 : \mathfrak{p}_{x,s-r}(-2) \rightarrow \mathfrak{p}_{x,s+r}(2)$$

is injective.

Let  $Z \in \mathfrak{p}_{x,s+r}(2) \subset \mathfrak{g}_{x,s+r}(2)$ . By [[9], 4.3.1], there is an element  $Z' \in \mathfrak{g}_{x,s-r}(-2)$  such that  $(\text{ad}(X)^2)(Z') = Z$ . By Lemma 4.8, we may write  $Z' = Z'_+ + Z'_-$ , where  $Z'_+ \in \mathfrak{h}_{x,s-r}$  and  $Z'_- \in \mathfrak{p}_{x,s-r}$ . By the last line in [[9], Section 4.3], the projection  $\mathfrak{g} \rightarrow \mathfrak{g}(i)$  preserves depth, so we let  $W$  denote the projection of  $Z'_-$  to the  $(-2)$  weight space. Then, since  $Z \in \mathfrak{p}_{x,s+r}(2)$ , we have  $(\text{ad}(X)^2)(W) = Z$ . Thus, the map is surjective.  $\square$

**Corollary 5.14.** *Suppose Hypotheses 5.3 and 5.7 hold. If  $X \in \mathfrak{p}_{x,r}(2)$  is a lift of  $e$ , then there are lifts  $Y \in \mathfrak{p}_{x,-r}$  of  $f$  and  $H \in \mathfrak{h}_{x,0}$  of  $h$  such that  $\{Y, H, X\}$  is a normal  $\mathfrak{sl}_2(k)$ -triple in  $\mathfrak{g}$ .*

*Proof.* Let  $X \in \mathfrak{p}_{x,r}(2)$  be a lift of  $e$ . By the previous lemma,  $\text{ad}(X)^2 : \mathfrak{p}_{x,-r}(-2) \rightarrow \mathfrak{p}_{x,r}(2)$  is surjective, so there exists an element  $Y \in \mathfrak{p}_{x,-r}(-2)$  with  $\text{ad}(X)^2(Y) = -2X$ . By the proof of [[9], Lemma 4.3.1],  $\text{ad}(e)^2 : \mathfrak{g}_x(-2) \rightarrow \mathfrak{g}_x(2)$  is injective, so since  $\text{ad}(e)^2(f) = -2e$  and  $\text{ad}(e)^2(\bar{Y} - f) = 0$ , we have that  $Y$  is a lift of  $f$ . Set  $H = [X, Y]$ . By a computation,  $[H, X] = 2X$ , so in order to show that  $\{Y, H, X\}$  is our desired  $\mathfrak{sl}_2(k)$ -triple, we must verify that  $[H, Y] = -2Y$ .

By [[6], Theorem 5.3.2], there exists some  $Y' \in \mathfrak{g}$  which completes  $\{H, X\}$  to an  $\mathfrak{sl}_2(k)$ -triple. By projecting  $Y'$  to  $\mathfrak{p}(-2)$ , we can assume it lies in this weight space. Now  $\text{ad}(X)^2(Y') = \text{ad}(X)^2(Y)$ , so by Lemma 5.13, since  $\text{ad}(X)^2$  is injective, we have  $Y = Y'$ .  $\square$

### 5.3 One-parameter subgroups

We now fix a one-parameter subgroup  $\lambda \in \mathbf{X}_*^k(\mathbf{H})$ . The following material is obtained from results in [[9], Section 4.4].

Fix an element  $X \in \mathcal{N} \cap \mathfrak{p}$ . Suppose Hypothesis 5.9 holds. Then, there exists a normal  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$  completing  $X$  and a homomorphism  $\varphi : \mathbf{SL}_2 \rightarrow \mathbf{G}$  so that  $H = d\varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$  and  $Y = d\varphi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)$ . Note that such a map is  $\text{Gal}(K/k)$ -equivariant.

We will now exhibit a point  $y \in \mathcal{B}(H)$  such that  $Y \in \mathfrak{p}_{y,-r}$ ,  $H \in \mathfrak{h}_{y,0}$ , and  $X \in \mathfrak{p}_{y,r}$ . The argument given in the lemma below (excluding the last paragraph) is due to Gopal Prasad.

**Lemma 5.15.** *(Barbasch and Moy). Suppose Hypothesis 5.9 holds. There exists some  $x \in \mathcal{B}(H)$  such that  $Y, H, X \in \mathfrak{g}_{x,0}$ .*

*Proof.* (Gopal Prasad) Let  $J = \varphi(\mathbf{SL}_2(R_K)) \subset \mathbf{G}^\circ(K)$ . Then,  $B := (J \rtimes \text{Gal}(K/k))$  is a subgroup of the group of polysimplicial automorphisms of  $\mathcal{B}(\mathbf{G}, K)$ . Note that  $\text{Gal}(K/k)$  is a profinite group; in particular, it is compact and bounded. Thus,  $B$  is also bounded, so by [[21], 2.3.1], there exists a fixed point  $x' \in \mathcal{B}(\mathbf{G}, K)$  under the action of  $B$ . Let  $\mathcal{G}$  denote the smooth affine  $R$ -group scheme whose  $R_K$ -points form the group  $\text{stab}_{\mathbf{G}^\circ(K)}(x')$  and whose generic fiber is  $\mathbf{G}^\circ$ . Let  $L(\mathcal{G})$  denote the Lie algebra of  $\mathcal{G}$ , and let  $\mathcal{J}$  denote the  $R$ -group scheme associated to the parahoric subgroup  $\mathbf{SL}_2(R_K)$ . By [[5], 1.7.6],  $\varphi$  induces a map of  $R_K$ -schemes from  $\mathcal{J}$  to  $\mathcal{G}$ . Thus  $d\varphi(\mathfrak{sl}_2(R_K)) \subset \mathfrak{g}(K)_{x'}$ . Now, since  $x'$  is fixed by  $\text{Gal}(K/k)$ , we have  $Y, H, X \in \mathfrak{g}_{x'}$ .

We have shown that the set of  $B$ -fixed points  $\Omega := \mathcal{B}(\mathbf{G}, K)^B$  is nonempty. Since  $\{Y, H, X\}$  is a normal triple, we have  $d\varphi(\mathfrak{sl}_2(R)) \subset \mathfrak{g}_{\theta(x')}$ . In particular, by [[9], Corollary 4.5.5], we have  $\theta(x') \in \Omega$ , so  $\Omega$  is  $\theta$ -stable. In particular, since  $\Omega$  is convex and closed, and  $\langle \theta \rangle$  is a bounded group of isometries, there exists a  $\theta$ -fixed point  $x \in \Omega$  for which  $Y, H, X \in \mathfrak{g}_x$ .  $\square$

Under Hypothesis 5.9, there is a homomorphism  $\varphi : \mathbf{SL}_2 \rightarrow \mathbf{G}$  with some nice properties with respect to  $\{Y, H, X\}$ . Let  $\lambda \in \mathbf{X}_*^k(\mathbf{G})$  be defined by  $\lambda(t) = \varphi \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)$ .

**Definition 5.16.** *The one-parameter subgroup  $\lambda$  described above is said to be adapted to the  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$ .*

**Remark 5.17.** *In the preliminaries of Section 2.1, we declared that an element  $X \in \mathfrak{p}$  is nilpotent provided that there exists some one-parameter subgroup  $\mu \in \mathbf{X}_*^k(\mathbf{G})$  such that*

$$\lim_{t \rightarrow 0} \mu(t) X = 0.$$

*However, assuming Hypothesis 5.9 is valid, we can give an alternate characterization of nilpotence which coincides with this notion. Namely, suppose  $X$  lies in  $\mathcal{N} \cap \mathfrak{p}$ , and suppose  $\{Y, H, X\}$  is a normal  $\mathfrak{sl}_2(k)$ -triple completing  $X$ . By Jacobson-Morosov, there exists some one-parameter subgroup  $\lambda$  for which  $\lambda(t)X = t^2X$ , for  $t \in k^\times$ . Since  $\{Y, H, X\}$  is normal,  $H$  is  $\theta$ -fixed; in particular, we may assume  $\lambda$  is fixed by  $\theta$ . Thus, under Hypothesis 5.9,  $X$  lies in  $\mathcal{N} \cap \mathfrak{p}$  if and only if there exists some one-parameter subgroup in  $\mathbf{X}_*^k(\mathbf{H})$  which annihilates  $X$  in the limit described above.*

As noted in the remark above, if  $\{Y, H, X\}$  is normal, we may assume  $\lambda \in \mathbf{X}_*^k(\mathbf{H})$ . Define  $M = C_{\mathbf{G}^\circ(k)}(\lambda)$ .

**Corollary 5.18.** *Suppose Hypotheses 5.7 and 5.9 hold. There exists some  $y \in \mathcal{B}(H)$  such that  $Y \in \mathfrak{p}_{y,-r}$ ,  $X \in \mathfrak{p}_{y,r}$ , and  $H \in \mathfrak{h}_{y,0}$ .*

*Proof.* Together, Hypotheses 5.7 and 5.9 imply that the residue field  $\mathfrak{f}$  has cardinality greater than 3. By Lemma 5.15, there is an element  $x \in \mathcal{B}(H)$  such that  $Y, H, X \in \mathfrak{g}_x$ . Since  $\lambda(R^\times) \subset J$  as in the proof of Lemma 5.15, we know that the point  $x$  is fixed by  $\lambda(R^\times)$ . In particular, by [[9], Corollary 4.4.2],  $x$  lies in  $\mathcal{B}(M)$ . Choose an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  which contains  $x$ . Since  $\lambda$  lies in the center of  $M$ ,  $\lambda$  acts on every apartment in  $\mathcal{B}(M)$  by translation. Using this fact, define  $y = x + \frac{r}{2} \cdot \lambda \in \mathcal{A}$ . By Lemma 5.15,  $X \in \mathfrak{p}_{x,0}$ , so we write  $X = \sum_\psi X_\psi$ , where  $X_\psi \in \mathfrak{g}_\psi$ , for  $\psi(x) \geq 0$ . For all such  $\psi$  such that  $X_\psi \neq 0$ , we have  $\langle \lambda, \dot{\psi} \rangle = 2$  since  $\lambda$  acts by squares on  $X$  by Hypothesis 5.3. For any such  $\psi$ , we have

$$\psi(y) = \psi(x) + \frac{r}{2} \langle \lambda, \dot{\psi} \rangle \geq r.$$

Therefore,  $X$  lies in  $\mathfrak{p}_{y,r}$ . By a similar argument,  $H \in \mathfrak{h}_{y,0}$  and  $Y \in \mathfrak{p}_{y,-r}$ .  $\square$

## 6 The parametrization

Fix  $r \in \mathbb{R}$ . We now discuss the notion of the building set associated to an  $\mathfrak{sl}_2(k)$ -triple, so we assume that Hypotheses 5.7 and 5.9 hold. We follow the discussion in [[9], Section 5]. Fix  $Z \in \mathcal{N}^-$  and  $s \in \mathbb{R}$ .

### 6.1 The building set

**Definition 6.1.**

$$\mathcal{B}(Z, s) := \{z \in \mathcal{B}(G) \mid Z \in \mathfrak{g}_{z,s}\}.$$

From [[9]], we know that  $\mathcal{B}(Z, s)$  is nonempty, convex and closed.

**Definition 6.2.**

$$\mathcal{B}_\theta(Z, s) := \mathcal{B}(Z, s) \cap \mathcal{B}(H).$$

Corollary 5.18 tells us that  $\mathcal{B}_\theta(Z, s)$  is nonempty. From Definition 6.2, we see that if  $x \in \mathcal{B}_\theta(Z, s)$ , then  $F_\theta^*(x) \subset \mathcal{B}_\theta(Z, s)$ . In particular,  $\mathcal{B}_\theta(Z, s)$  is the union of generalized  $(s, \theta)$ -facets of  $\mathcal{B}(H)$ . Since  $\mathcal{B}(Z, s)$  and  $\mathcal{B}(H)$  are convex,  $\mathcal{B}_\theta(Z, s)$  is also convex.

**Lemma 6.3.**  $\mathcal{B}_\theta(Z, s)$  is closed.

*Proof.* This follows from the fact that  $\mathcal{B}(H)$  and  $\mathcal{B}(Z, s)$  are closed. □

Fix a (possibly trivial) normal  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$  in  $\mathfrak{g}$ .

**Definition 6.4.** Define

$$\mathcal{B}(Y, H, X) := \mathcal{B}(X, r) \cap \mathcal{B}(Y, -r).$$

**Definition 6.5.** Define

$$\mathcal{B}_\theta(Y, H, X) := \mathcal{B}(Y, H, X)^\theta.$$

By [[9], Remark 5.1.5],  $\mathcal{B}(Y, H, X)$  is convex. In particular,  $\mathcal{B}_\theta(Y, H, X)$  is a closed, convex set which is the union of generalized  $(r, \theta)$ -facets.

**Lemma 6.6.** Suppose  $F_{1,\theta}^*, F_{2,\theta}^*$  are maximal generalized  $(r, \theta)$ -facets in  $\mathcal{B}_\theta(Y, H, X)$ . Then,  $F_{1,\theta}^*$  and  $F_{2,\theta}^*$  are strongly  $r$ -associated.

*Proof.* Let  $x_i \in F_{i,\theta}^*$ , for  $i = 1, 2$ , and let  $\mathcal{A}$  be an apartment of  $\mathcal{B}(H)$  containing  $x_1$  and  $x_2$ . If  $x_1 \notin A(\mathcal{A}, F_{2,\theta}^*)$ , then, since  $\mathcal{B}_\theta(Y, H, X)$  is convex, there is some generalized  $(r, \theta)$ -facet in  $\mathcal{B}_\theta(Y, H, X)$  of strictly larger dimension than  $F_{2,\theta}^* \cap \mathcal{A}$ , a contradiction. Thus,  $A(\mathcal{A}, F_{1,\theta}^*) \subset A(\mathcal{A}, F_{2,\theta}^*)$ , and similarly,  $A(\mathcal{A}, F_{2,\theta}^*) \subset A(\mathcal{A}, F_{1,\theta}^*)$ . □

We now suppose that Hypotheses 5.7, 5.9, and 5.11 hold. Fix  $X \in \mathcal{N}^- \setminus \{0\}$  and  $r \in \mathbb{R}$ . Suppose that  $\{Y, H, X\}$  is a normal  $\mathfrak{sl}_2(k)$ -triple completing  $X$  and that  $\lambda \in \mathbf{X}_*^k(\mathbf{H})$  is adapted to  $\{Y, H, X\}$ . Fix  $x \in \mathcal{B}_\theta(Y, H, X)$ . We would like for  ${}^H X$  to be the unique nilpotent  $H$ -orbit in  $\mathfrak{p}$  of minimal dimension which intersects the coset  $X + \mathfrak{p}_{x,r+}$  nontrivially. The next lemma gives us a decomposition of the coset  $X + \mathfrak{p}_{x,r+}$  up to conjugacy by  $H_x^+$ . Recall that the one-parameter subgroup  $\lambda$  induces a grading on the Lie algebra of  $\mathfrak{g}$  as noted in the beginning of Section 5.2. For the following lemma, we imitate the argument in [[9], 5.2.1].

**Lemma 6.7.** *Assume Hypotheses 5.7, 5.9, and 5.11 hold. Then*

$$H_x^+(X + C_{\mathfrak{p}_{x,r^+}}(Y)) = X + \mathfrak{p}_{x,r^+}.$$

*Proof.* “ $\subset$ ”: By [[1], Prop 1.4.3],  $H_x^+$  induces the trivial action on  $V_{x,r}$ .

“ $\supset$ ”: From Hypothesis 5.3, we know that as a representation of  $\langle Y, H, X \rangle$ ,  $\mathfrak{g}$  decomposes into a direct sum of irreducible  $\langle Y, H, X \rangle$ -modules with highest weight at most  $p - 3$ . Thus, we can write

$$\mathfrak{g} = \bigoplus_{\rho \in \mathbb{Z}_{\geq 0}} \mathfrak{g}_\rho$$

where  $\mathfrak{g}_\rho$  is the isotypic component of  $\langle Y, H, X \rangle$ -modules in  $\mathfrak{g}$  with highest weight  $\rho$ . In other words,  $\mathfrak{g}_\rho$  is the direct sum of irreducible  $\langle Y, H, X \rangle$ -submodules of  $\mathfrak{g}$  of dimension  $\rho + 1$ . Let  $\mathfrak{g}(\rho, i) := \mathfrak{g}_\rho \cap \mathfrak{g}(i)$ , and let  $\mathfrak{p}(\rho, i) := \mathfrak{g}(\rho, i) \cap \mathfrak{p}$ . Then,  $\mathfrak{g}(i) = \bigoplus_\rho \mathfrak{g}(\rho, i)$  and thus  $\mathfrak{g} = \bigoplus_{\rho, i} \mathfrak{g}(\rho, i)$ . Also, note that we have that  $C_{\mathfrak{g}}(X) = \bigoplus_{i \geq 0} \mathfrak{g}(i, i)$  since  $\text{ad}(X)(\mathfrak{g}(i)) = \mathfrak{g}(i+2)$ , and  $\mathfrak{g}(i, i)$  is the sum of  $i$ -weight spaces of all irreducible  $\langle Y, H, X \rangle$ -submodules with highest weight  $i$ . Similarly, we have  $C_{\mathfrak{g}}(Y) = \bigoplus_{i \geq 0} \mathfrak{g}(i, -i)$ . It follows that  $C_{\mathfrak{p}}(Y) = \bigoplus_{i \geq 0} \mathfrak{p}(i, -i)$  and  $C_{\mathfrak{p}}(X) = \bigoplus_{i \geq 0} \mathfrak{p}(i, i)$ . Define  $\mathfrak{g}_{x,s}(\rho, i) := \mathfrak{g}_{x,s} \cap \mathfrak{g}(\rho, i)$ . By the proof of [[9], Lemma 5.2.1], since the projection  $\mathfrak{g} \rightarrow \mathfrak{g}(i)$  preserves depth, we have the following decompositions:

$$\mathfrak{g}_{x,s}(i) = \bigoplus_{\rho \in \mathbb{Z}_{\geq 0}} \mathfrak{g}_{x,s}(\rho, i)$$

and

$$\mathfrak{g}_{x,s} = \bigoplus_{\rho, i} \mathfrak{g}_{x,s}(\rho, i).$$

From the first decomposition, using Proposition 4.8 it follows that

$$\mathfrak{h}_{x,s}(i) = \bigoplus_{\rho \in \mathbb{Z}_{\geq 0}} \mathfrak{h}_{x,s}(\rho, i),$$

and

$$\mathfrak{p}_{x,s}(i) = \bigoplus_{\rho \in \mathbb{Z}_{\geq 0}} \mathfrak{p}_{x,s}(\rho, i).$$

We claim that the following decompositions hold:

$$(\dagger) \text{ For } i > 0, \mathfrak{p}_{x,s}(i) = \text{ad}(X)(\mathfrak{h}_{x,s-r}(i-2))$$

and

$$(\dagger\dagger) \text{ For } i \leq 0, \mathfrak{p}_{x,s}(i) = \mathfrak{p}_{x,s}(-i, i) + \text{ad}(X)(\mathfrak{h}_{x,s-r}(i-2)).$$

The first decomposition  $(\dagger)$  is obtained using decompositions  $\mathfrak{h}_{x,s}(i) = \bigoplus_{\rho \in \mathbb{Z}_{\geq 0}} \mathfrak{h}_{x,s}(\rho, i)$  and  $\mathfrak{p}_{x,s}(i) = \bigoplus_{\rho \in \mathbb{Z}_{\geq 0}} \mathfrak{p}_{x,s}(\rho, i)$  and the fact that  $\mathfrak{p}(\rho, i) = \{Z \in \mathfrak{p}(i) \mid (\text{ad}(X) \circ \text{ad}(Y))(Z) = j(\rho, i) \cdot Z\}$ , which can be found in [[9], 5.2.1].

The second decomposition  $(\dagger\dagger)$  results from the fact that for  $Z \in \mathfrak{p}_{x,s}(i), i \leq 0$ ,  $Z$  may lie in  $C_{\mathfrak{p}}(Y)$ . We also use the fact that  $\mathfrak{p}_{x,s}(i) = \bigoplus_{\rho \in \mathbb{Z}_{\geq 0}} \mathfrak{p}_{x,s}(\rho, i)$ .

Now, summing over all  $i$ , we obtain

$$(\star) \mathfrak{p}_{x,s} = C_{\mathfrak{p}_{x,s}}(Y) + \text{ad}(X)(\mathfrak{h}_{x,s-r}).$$

Let  $Z \in \mathfrak{p}_{x,r+}$ . We will show the existence of elements  $h \in H_x^+$  and  $C \in C_{\mathfrak{p}_{x,r+}}(Y)$  such that  ${}^h(X + C) = X + Z$ . First, let  $h_0 = 1$  and  $C_0 = 0$ . Now, choose  $s_1 \in \mathbb{R}$  with  $\mathfrak{p}_{x,r+} = \mathfrak{p}_{x,s_1} \neq \mathfrak{p}_{x,s_1^+}$ . Using  $(\star)$ , we can write  $Z = C'_1 + \text{ad}(X)(P_1)$  where  $C'_1 \in C_{\mathfrak{p}_{x,s_1}}(Y)$  and  $P_1 \in \mathfrak{h}_{x,(s_1-r)}$ . Applying Hypothesis 5.11 with  $s = s_1 - r$  and  $t = s_1$ , there exists a map  $\phi_x : \mathfrak{h}_{x,s_1-r} \rightarrow H_{x,s_1-r}$  such that

$$(1) \phi_x(-P_1)(X + C_0 + C'_1) = X + C'_1 + \text{ad}(X)(P_1) \bmod \mathfrak{p}_{x,s_1^+}.$$

Set  $h'_1 = \phi_x(-P_1)$ . Rewriting the above equation, we have  ${}^{h'_1 h_0}(X + C_0 + C'_1) = X + Z - Z_1$  for some  $Z_1 \in \mathfrak{p}_{x,s_1^+}$ . Let  $h_1 = h'_1 h_0$  and  $C_1 = C_0 + C'_1$ . Now, fix an element  $s_2 > s_1$  such that  $\mathfrak{p}_{x,s_1^+} = \mathfrak{p}_{x,s_2} \neq \mathfrak{p}_{x,s_2^+}$ . Continuing as in the previous case, from  $(\star)$ , we write  $Z_1 = C'_2 + \text{ad}(X)(P_2)$  where  $C'_2 \in C_{\mathfrak{p}_{x,s_2}}(Y)$  and  $P_2 \in \mathfrak{h}_{x,s_2-r}$ . Applying Hypothesis 5.11 and (1), there exists a map  $\phi_x$  such that

$$\begin{aligned} \phi_x(-P_2)(X + C_1 + C'_2) &= {}^{h'_2}(X + Z - Z_1 + C'_2) \bmod \mathfrak{p}_{x,s_2^+} \\ &= X + Z - Z_1 + C'_2 + \text{ad}(X)P_2 \bmod \mathfrak{p}_{x,s_2^+} \\ &= X + Z - Z_2. \end{aligned}$$

where  $Z_2 \in \mathfrak{p}_{x,s_2^+}$  and  $h'_2 = \phi_x(-P_2)$ . Set  $h_2 = h'_2 h_1$  and  $C_2 = C_1 + C'_2$ . Proceeding as above, we obtain a strictly increasing sequence  $\{s_i\}$  with  $s_1 > r$  such that  $h_n \in H_x^+$  and  $h'_n \in H_{x,(s_n-r)}$ . Moreover, we have elements  $C_n = C_{n-1} + C'_n \in C_{\mathfrak{p}_{x,r+}}(Y)$  such that  $C'_n \in C_{\mathfrak{p}_{x,s_n}}(Y)$  and

$${}^{h_n}(X + C_n) = X + Z \bmod \mathfrak{p}_{x,s_n^+}.$$

Now set  $h = \lim_{n \rightarrow \infty} h_n$  and  $C = \lim_{n \rightarrow \infty} C_n$ . Clearly, these elements lie in  $H_x^+$  and  $C_{\mathfrak{p}_{x,r+}}(Y)$ , respectively. By construction, we have  ${}^h(X + C) = X + Z$ .  $\square$

**Lemma 6.8.** *Suppose Hypothesis 5.9 holds. Then*

$$(X + C_{\mathfrak{p}}(Y)) \cap {}^H X = \{X\}.$$

*Proof.* The result [ [23], V.7 (9)] tells us that  $(X + C_{\mathfrak{g}}(Y)) \cap {}^G X = \{X\}$ , so, consequently,  $(X + C_{\mathfrak{p}}(Y)) \cap {}^H X \subset \{X\}$ . Since  $\{X\}$  is clearly contained in the intersection, the result follows.  $\square$

**Corollary 6.9.** *Suppose Hypotheses 5.7, 5.9, and 5.11 hold. Then*

$$(X + \mathfrak{p}_{x,r+}) \cap {}^H X = {}^{H_x^+} X.$$

*Proof.* “ $\supset$ ”: This inclusion follows from the first part of the proof of Lemma 6.7.

“ $\subset$ ”: Using Lemma 6.7, we have  $X + \mathfrak{p}_{x,r+} = {}^{H_x^+} (X + C_{\mathfrak{p}_{x,r+}}(Y))$ . Thus, if  $Z$  lies in  $(X + \mathfrak{p}_{x,r+}) \cap {}^H X$ , then there exist elements  $h_1 \in H_x^+$ ,  $h_2 \in H$  and  $Z_1 \in C_{\mathfrak{p}_{x,r+}}(Y)$  such that  $Z = {}^{h_1}(X + Z_1) = {}^{h_2} X$ . Thus  ${}^{h_1^{-1}} Z = X + Z_1 = {}^{h_1^{-1} h_2} X$ , so by Corollary 6.8, we have



$$X + Z_1 \in (X + C_{\mathfrak{p}}(Y)) \cap {}^H X = \{X\}$$

Thus,  $Z_1 = 0$ .

□

**Definition 6.10.** We denote by  $\mathcal{O}_{\theta}(0)$  the set of all nilpotent  $H$ -orbits in  $\mathfrak{p}$ .

In the statement of the following corollary, if  $\mathcal{O}_{\theta}$  is an element of  $\mathcal{O}_{\theta}(0)$ , we let  $\overline{\mathcal{O}_{\theta}}$  denote the  $p$ -adic closure of  $\mathcal{O}_{\theta}$ .

**Corollary 6.11.** Suppose Hypotheses 5.7, 5.9, and 5.11 hold. If  $\mathcal{O}_{\theta} \in \mathcal{O}_{\theta}(0)$  such that

$$(X + \mathfrak{p}_{x,r+}) \cap \mathcal{O}_{\theta} \neq \emptyset,$$

then  ${}^H X \subset \overline{\mathcal{O}_{\theta}}$ .

*Proof.* (J.L. Waldspurger) Let  $Z \in (X + \mathfrak{p}_{x,r+}) \cap \mathcal{O}_{\theta}$ . Then by Lemma 6.7, there exist elements  $h \in H_x^+$  and  $C \in C_{\mathfrak{p}_{x,r+}}(Y)$  such that  ${}^h(X + C) = Z \in \mathcal{O}_{\theta}$ . Inverting  $h$ , we have  $X + C \in \mathcal{O}_{\theta}$ , which is therefore nilpotent. By Jacobson-Morosov, there exists a one-parameter subgroup  $\mu \in \mathbf{X}_*^k(\mathbf{H})$  such that  $\mu^{(t)}(X + C) = t^2 \cdot (X + C)$  for all  $t \in k^{\times}$ . Recall that we let  $\lambda$  denote the one-parameter subgroup adapted to  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$ . In particular,  $\lambda^{(t)}X = t^2 \cdot X$ , for  $t \in k^{\times}$ . Moreover, from the proof of Lemma 6.7, we know that  $C_{\mathfrak{p}}(Y) = \oplus_{i \geq 0} \mathfrak{p}(i, -i)$ , so, in particular,  $C \in \oplus_{i \leq 0} \mathfrak{p}(i)$ . Thus,

$$\lim_{t \rightarrow 0} \lambda^{(t)^{-1} \mu^{(t)}}(X + C) = \lim_{t \rightarrow 0} \lambda^{(t)^{-1}} t^2 (X + C) = X + \lim_{t \rightarrow 0} \lambda^{(t)^{-1}} C = X.$$

□

**Corollary 6.12.** Suppose Hypotheses 5.7, 5.9, and 5.11 hold. Choose  $F_{\theta}^* \in \mathcal{F}_{\theta}(r)$  such that  $F_{\theta}^* \subset \mathcal{B}_{\theta}(Y, H, X)$ . If  $\mathcal{O}_{\theta} \in \mathcal{O}_{\theta}(0)$  such that

$$(X + \mathfrak{p}_{F_{\theta}^*}^+) \cap \mathcal{O}_{\theta} \neq \emptyset,$$

then  ${}^H X \subset \overline{\mathcal{O}_{\theta}}$ .

*Proof.* If  $x \in F_{\theta}^*$ , then  $X \in \mathfrak{p}_{x,r} = \mathfrak{p}_{F_{\theta}^*}$ . Thus, by Corollary 6.11,  ${}^H X \subset \overline{\mathcal{O}_{\theta}}$ . □

**Definition 6.13.** Define  $I_r^n := \{(F_{\theta}^*, v) \in I_r \mid v \text{ is degenerate in } V_{F_{\theta}^*}^-\}$ .

Suppose  $(F_{\theta}^*, e) \in I_r^n$ . Let  $x \in F_{\theta}^*$ . If  $e$  is trivial, then we call the normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple completing  $e$  (in  $V_{x,-r}^- \times V_{x,0}^+ \times V_{x,r}^-$ ) the trivial  $\mathfrak{sl}_2(\mathfrak{f})$ -triple. Similarly, given a trivial  $\mathfrak{sl}_2(\mathfrak{f})$ -triple, we declare that the  $\mathfrak{sl}_2(k)$ -triple lifting our  $\mathfrak{sl}_2(\mathfrak{f})$ -triple is the trivial  $\mathfrak{sl}_2(k)$ -triple.

**Lemma 6.14.** Suppose all hypotheses from Section 5 hold, and let  $(F_{\theta}^*, e) \in I_r^n$ .

1. Fix  $x \in F_{\theta}^*$ . There exists a normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple  $(f, h, e) \in V_{x,-r}^- \times V_{x,0}^+ \times V_{x,r}^-$  completing  $e$  and a normal  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$  which lifts  $\{f, h, e\}$ .

2. For any  $x \in F_\theta^*$ , for any normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple  $(f, h, e) \in V_{x,-r}^- \times V_{x,0}^+ \times V_{x,r}^-$  completing  $e$ , and for any normal  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$  which lifts  $\{f, h, e\}$ , we have  $F_\theta^* \subset \mathcal{B}_\theta(Y, H, X)$ , and  ${}^H X$  is the unique nilpotent  $H$ -orbit in  $\mathfrak{p}$  of minimal dimension which intersects the coset  $e$  nontrivially.

*Proof.* If  $e$  is trivial, then all the conclusions are obvious. Assume  $e$  is nontrivial. By Hypothesis 5.3, there exist elements  $H \in \mathfrak{h}_{x,0}$  and  $Y \in \mathfrak{p}_{x,-r}$  such that the image  $\{f, h, e\}$  of the triple  $\{Y, H, X\}$  forms a normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple in  $\bar{\mathfrak{g}}_x$ . By Corollary 5.14, there is a normal  $\mathfrak{sl}_2(k)$ -triple lifting  $\{f, h, e\}$ .

For (2), suppose  $x \in F_\theta^*$ , the triple  $\{f, h, e\} (\subset V_{x,-r}^- \times V_{x,0}^+ \times V_{x,r}^-)$  is a normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple in  $\bar{\mathfrak{g}}_x$  completing  $e$ , and  $\{Y, H, X\}$  is a normal  $\mathfrak{sl}_2(k)$ -triple which lifts  $\{f, h, e\}$ . Note that since  $x \in F_\theta^*$ , we have  $F_\theta^* \subset \mathcal{B}_\theta(Y, H, X)$  since  $X \in \mathfrak{p}_{x,r}$  and  $Y \in \mathfrak{p}_{x,-r}$ . Now, if  ${}^H Z$  is a nilpotent  $H$ -orbit such that  $(X + \mathfrak{p}_{x,r+}) \cap {}^H Z \neq \emptyset$ , then by Corollary 6.12, we have  ${}^H X \subset \overline{{}^H Z}$ . Thus, since the closure of any  $H$ -orbit is the union of the orbit itself and orbits of strictly smaller dimension,  ${}^H X$  is the unique nilpotent  $H$ -orbit in  $\mathfrak{p}$  of minimal dimension which intersects the coset  $e$  nontrivially.  $\square$

**Definition 6.15.** Suppose all the hypotheses of Section 5 hold. For  $(F_\theta^*, e) \in I_r^n$ , let  $\mathcal{O}_\theta(F_\theta^*, e)$  denote the unique nilpotent  $H$ -orbit of minimal dimension which intersects the coset  $e$  nontrivially.

**Remark 6.16.** If  $h \in H$  and  $(F_\theta^*, e) \in I_r^n$ , then it is clear from Definition 6.15 that  $\mathcal{O}_\theta(hF_\theta^*, {}^h e) = \mathcal{O}_\theta(F_\theta^*, e)$ .

**Lemma 6.17.** Suppose all the hypotheses of Section 5 hold. The map  $\varphi : I_r^n \rightarrow \mathcal{O}_\theta(0)$  defined by  $(F_\theta^*, e) \mapsto \mathcal{O}_\theta(F_\theta^*, e)$  induces a well-defined map from  $I_r^n / \sim$  to  $\mathcal{O}_\theta(0)$ .

*Proof.* Suppose  $(F_{i,\theta}^*, e_i) \in I_r^n$  and  $(F_{1,\theta}^*, e_1) \sim (F_{2,\theta}^*, e_2)$ . Suppose  $e_i \in V_{F_{i,\theta}^*}^-$  is nontrivial. Choose  $x_i \in C(F_{i,\theta}^*)$ . Since  $(F_{1,\theta}^*, e_1) \sim (F_{2,\theta}^*, e_2)$ , there exists an element  $h \in H$  and an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  such that

1.  $A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, hF_{2,\theta}^*) \neq \emptyset$ ,
2.  $e_1 = {}^h e_2$  in  $V_{F_{1,\theta}^*}^- = V_{hF_{2,\theta}^*}^-$ .

As a result of Remark 6.16, we assume now that  $h = 1$ . Let  $\mathbf{S}$  be the maximal  $k$ -split torus of  $\mathbf{H}$  corresponding to the apartment  $\mathcal{A}$ . Let  $\mathbf{S}$  denote the maximal  $\mathfrak{f}$ -split torus inside  $\mathbf{H}_{x_1}$  corresponding to  $\mathbf{S}$ . By the previous lemma, we can complete  $e_1$  to a normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple  $(f, h, e_1) \in V_{x_1,-r}^- \times V_{x_1,0}^+ \times V_{x_1,r}^-$ . Suppose that  $\bar{\lambda} \in \mathbf{X}_*^{\mathfrak{f}}(\mathbf{H}_{x_1})$  is adapted to this triple. Since any one-parameter subgroup is contained in some maximal  $\mathfrak{f}$ -split torus, there exists some  $h' \in H_{x_1}$  such that  ${}^{h'} \bar{\lambda} \in \mathbf{X}_*^{\mathfrak{f}}(\mathbf{S})$ . By Lemma 4.40, there is an element  $h'' \in H_{x_1} \cap H_{x_2}$  such that its image in  $\text{Aut}_{\mathfrak{f}}(V_{F_{i,\theta}^*}^-)$  coincides with the image of  $h'$ . In other words, we have  $\text{Ad}(h'')|_{V_{F_{i,\theta}^*}^-} = \text{Ad}(h')|_{V_{F_{i,\theta}^*}^-}$ . In summary, we have

$${}^{h'} e_1 = {}^{h''} e_1 = {}^{h''} e_2 \text{ in } V_{F_{1,\theta}^*}^- = V_{F_{1,\theta}^*}^- = V_{F_{2,\theta}^*}^-.$$

Now let  $\lambda \in \mathbf{X}_*^k(\mathbf{S})$  be a lift of  ${}^{h'} \bar{\lambda}$ . As usual, we have a grading of  $\mathfrak{g}$  under the action of  $\lambda$ . As in the proof of Lemma 6.7, we also have the decomposition

$$\mathfrak{g}_{F_{i,\theta}^*}^{F_{i,\theta}^*} = \bigoplus_j \mathfrak{g}_{F_{i,\theta}^*}^{F_{i,\theta}^*}(j).$$

By Lemma 4.37, there is a lift  $X \in \mathfrak{p}_{F_{1,\theta}^*}^*(2) \cap \mathfrak{p}_{F_{2,\theta}^*}^*(2)$  of  $h'' e_i$ . Note that  $X$  lifts  $h' e_i$  and  $h''^{-1} X$  lifts  $e_i$ . We apply Corollary 5.14 and Lemma 6.14 to conclude

$$\mathcal{O}_\theta(F_{i,\theta}^*, e_i) = \mathcal{O}_\theta(F_{i,\theta}^*, h'' e_i) = {}^H X.$$

□

Now, in order to obtain a bijection between depth  $r$  cosets and nilpotent  $H$ -orbits, we have to shrink  $I_r^n / \sim$ . We do this by restricting to *noticed* orbits in  $\mathfrak{p}_{F_\theta^*}^* / \mathfrak{p}_{F_\theta^*}^+$ .

**Definition 6.18.** Define  $I_r^d \subset I_r^n$  to be those pairs  $(F_\theta^*, e) \in I_r^n$  such that for any  $x \in F_\theta^*$ , for any normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple  $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$  completing  $e$ , and for any normal  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$  in  $\mathfrak{g}$  which lifts  $\{f, h, e\}$ , we have that  $F_\theta^*$  is a maximal generalized  $(r, \theta)$ -facet in  $\mathcal{B}_\theta(Y, H, X)$ .

In order to simplify notation below, we will refer to a generalized  $(0, \theta)$ -facet as a generalized  $\theta$ -facet.

**Remark 6.19.** In the case when  $r = 0$ , we can give an alternate characterization of noticed nilpotent orbits inside a Lie algebra which is analogous to the definition in [[18], Definition 2.1]. In particular, if a pair  $(F_\theta^*, e)$  lies in  $I_0^d$ , then  $C_{V_{F_\theta^*}^+}(e)$  does not contain certain non-central semisimple elements. We give a precise formulation below.

Suppose  $(F_\theta^*, e) \in I_0^n$ , and let  $x \in F_\theta^*$ . In this paragraph, we let  $\mathbb{L}_x$  denote the Lie algebra of  $\mathbb{G}_x$ . Under some restrictions on the characteristic of  $\mathfrak{f}$ , it is shown in [[6], Proposition 5.7.4] that if  $\{f, h, e\}$  is an  $\mathfrak{sl}_2(\mathfrak{f})$ -triple in  $\mathbb{L}_x$  completing  $e$ , then  $C_{\mathbb{L}_x}(e)$  is a subalgebra of  $\mathbb{L}_x$  of the form  $\mathfrak{c} \oplus \mathfrak{u}$ , where  $\mathfrak{c}$  is a reductive subalgebra which centralizes the triple  $\{f, h, e\}$  and  $\mathfrak{u}$  is a nilpotent ideal in  $C_{\mathbb{L}_x}(e)$ . In particular, we may consider the  $\mathfrak{f}$ -rank of  $\mathfrak{c}$  in the sense of [[3], 21.1]. By [[6], Proposition 5.9.3], if  $\{f', h', e\}$  is another  $\mathfrak{sl}_2(\mathfrak{f})$ -triple in  $\mathbb{L}_x$  completing  $e$ , and if  $\mathfrak{c}'$  is defined relative to  $\{f', h', e\}$ , then  $\mathfrak{c}$  and  $\mathfrak{c}'$  are conjugate by an element of  $C_{\mathbb{G}_x(\mathfrak{f})}(e)$ . In particular, the  $\mathfrak{f}$ -rank of  $\mathfrak{c}$  and  $\mathfrak{c}'$  are the same.

In the following proposition, we take the  $\mathfrak{f}$ -rank of  $C_{\mathbb{L}_x}(e)$  to mean the  $\mathfrak{f}$ -rank of the centralizer  $C_{\mathbb{L}_x}(\text{im } \phi)$ , where  $\phi : \mathfrak{sl}_2(\mathfrak{f}) \rightarrow V_{F_\theta^*}$  is an  $\mathfrak{f}$ -map whose image contains  $e$ .

**Proposition 6.20.** Let  $(F_\theta^*, e) \in I_0^n$ . Suppose Hypothesis 5.3 holds,  $\mathfrak{f}$  is finite, and  $p > 3(j-1)$ , where  $j$  is the Coxeter number as defined in [[6], Section 1.9]. The following are equivalent:

1.  $C_{V_{F_\theta^*}^+}(e) \cap [V_{F_\theta^*}, V_{F_\theta^*}]$  has  $\mathfrak{f}$ -rank equal to zero.

2.  $(F_\theta^*, e)$  lies in  $I_0^d$ .

*Proof.* (1)  $\Rightarrow$  (2) : Suppose there is some  $x \in F_\theta^*$  for which there exists a lift  $\{Y, H, X\} \subset V_{x,0}$  of an  $\mathfrak{sl}_2(\mathfrak{f})$ -triple in  $\bar{\mathfrak{g}}_x$  completing  $e$  such that  $F_\theta^*$  is not maximal in  $\mathcal{B}_\theta(Y, H, X)$ . Then, there exists some generalized  $\theta$ -facet  $C_\theta^*$  in  $\mathcal{B}_\theta(Y, H, X)$  containing  $F_\theta^*$  in its closure. In particular,

by [[9], Cor. 3.2.19], we may identify  $V_{C_\theta^*}$  with a Levi subalgebra of the parabolic subalgebra  $\mathfrak{g}_{C_\theta^*}/\mathfrak{g}_{F_\theta^*}^+$ , which are both  $\theta$ -stable by the proof of Proposition 4.8. Let  $\mathcal{A}$  be an apartment in  $\mathcal{B}(H)$  containing  $x$ , and let  $\mathbf{S}$  be the maximal  $k$ -split torus in  $\mathbf{H}$  corresponding to  $\mathcal{A}$ . By [[14], Proposition 5.4] and [[19], Theorem 1.9], we can choose some  $\theta$ -stable maximal  $k$ -split torus  $\mathbf{S}'$  containing  $\mathbf{S}$  such that  $\mathcal{A}(\mathbf{S}, k) \subset \mathcal{A}(\mathbf{S}', k)$ . Let  $\mathbf{S}$  denote the maximal  $\mathfrak{f}$ -split torus in  $\mathbf{H}_x$  corresponding to  $\mathbf{S}$ . In the notation of [[12], Lemma 3.3] (letting  $k = \mathfrak{f}$  and  $\lambda = \mu$ ), there exists some  $\mu \in \mathbf{X}_*^{\mathfrak{f}}(\mathbf{S})$  such that  $V_{C_\theta^*} = C_{V_{F_\theta^*}^*}(d\mu)$ . In particular, since  $C_\theta^* \subset \mathcal{B}_\theta(Y, H, X)$ , we have  $e \in V_{C_\theta^*}$ . Thus, there is a semisimple element in  $V_{F_\theta^*}^+$  which centralizes  $e$  and lies in the Lie algebra of some  $\mathfrak{f}$ -split torus in  $\mathbf{H}_x$ . Moreover, this element does not lie in the center of  $V_{F_\theta^*}^*$  since  $V_{C_\theta^*}$  is properly contained in  $V_{F_\theta^*}^*$ .

(2)  $\Rightarrow$  (1): We prove the contrapositive. Let  $x \in F_\theta^*$ . Suppose there is some element  $s \in C_{V_{F_\theta^*}^+}(e) \cap [V_{F_\theta^*}, V_{F_\theta^*}]$ , which lies in the Lie algebra of some  $\mathfrak{f}$ -split torus in  $\mathbf{H}_x$ . By [[3], 8.15(d)], we may assume there is some one-parameter subgroup  $\bar{\mu} \in \mathbf{X}_*^{\mathfrak{f}}(\mathbf{H}_x)$  for which  $s \in \text{im}(d\bar{\mu})$ . By [[14], Proposition 3.4.1] or [[12], Proposition 2.3], there is some  $\theta$ -stable maximal  $\mathfrak{f}$ -split torus  $\mathbf{T}$  in  $\mathbf{G}_x$  such that  $\bar{\mu} \in \mathbf{X}_*^{\mathfrak{f}}(\mathbf{T})$ . Since  $\bar{\mu}$  is  $\theta$ -fixed,  $\bar{\mu}$  determines a  $\theta$ -stable  $\mathfrak{f}$ -parabolic subgroup  $\mathbf{P}$  of  $\mathbf{G}_x$  which contains a  $\theta$ -stable  $\mathfrak{f}$ -Levi subgroup  $\mathbf{M} := C_{\mathbf{G}_x}(\bar{\mu})$ . Let  $\mathfrak{m}$  denote the Lie algebra of  $\mathbf{M}$ , which lies inside the Lie algebra of  $\mathbf{P}$ . By [[14], Proposition 5.4], we can choose a  $\theta$ -stable maximal  $k$ -split torus  $\mathbf{T}$  such that  $\mathcal{A}(\mathbf{T}, k)$  contains  $x$  and  $\mathcal{A}(\mathbf{T}, k)^\theta$  is an affine space whose dimension is equal to the dimension of a maximal  $k$ -split torus of  $\mathbf{H}$ . Using the embedding from [[19], Theorem 1.9], we must have that  $\mathcal{A}(\mathbf{T}, k)^\theta$  is an apartment of  $\mathcal{B}(H)$  containing  $x$ . The image of  $\mathbf{T}$  in  $\mathbf{G}_x$  is maximal  $\mathfrak{f}$ -split torus  $\mathbf{S}$  for which  $\mathbf{S}^\theta$  is a maximal  $\mathfrak{f}$ -split torus in  $\mathbf{H}_x$ . Thus, by conjugacy of maximal  $\mathfrak{f}$ -split tori in  $\mathbf{H}_x$ , we may assume that the image of  $\mathbf{T}$  in  $\mathbf{G}_x$  is  $\mathbf{T}$ . Using the identification of  $\mathbf{X}_*^k(\mathbf{T})$  with  $\mathbf{X}_*^{\mathfrak{f}}(\mathbf{T})$ , let  $\mu$  be a lift of  $\bar{\mu}$  in  $\mathbf{X}_*^k(\mathbf{T}) \otimes \mathbb{R}$ , let  $C$  be the first  $\theta$ -facet of  $\mathcal{A}(\mathbf{T}, k)$  encountered when moving from  $x$  to  $x + \mu$ , and let  $C_\theta^*$  be the generalized  $\theta$ -facet containing  $C$ . Then, the vector space of  $\mathfrak{f}$ -rational points of the Lie algebra of  $\mathbf{P}$  is of the form  $\mathfrak{g}_{C_\theta^*}/\mathfrak{g}_{F_\theta^*}^+$ . By our choice of  $\bar{\mu}$ , we have  $e \in \mathfrak{m} \subset \mathfrak{g}_{C_\theta^*}/\mathfrak{g}_{F_\theta^*}^+$ .

Let  $e = X' + \mathfrak{g}_{F_\theta^*}^+$ , for some representative  $X'$  which lies in  $\mathfrak{g}_{C_\theta^*}$ . If  $e$  is trivial, then we complete  $e$  to the trivial  $\mathfrak{sl}_2(\mathfrak{f})$ -triple inside  $V_{C_\theta^*}$  and let  $\{Y, H, X\}$  be the trivial  $\mathfrak{sl}_2(k)$ -triple in  $\mathfrak{g}_{C_\theta^*}$  lifting  $\{f, h, e\}$ . This shows that  $F_\theta^*$  cannot be maximal in  $\mathcal{B}_\theta(Y, H, X)$ .

Suppose  $e$  is nontrivial. We want to show that there exists some lift  $X$  of  $e$  in  $\mathfrak{g}_{C_\theta^*} \cap \mathcal{N}$ . Since  $e$  is degenerate in  $V_{F_\theta^*}$ , using the same argument as that given in the last paragraph of Lemma 5.2, there exists some  $X \in \mathfrak{g}_{C_\theta^*} \cap \mathcal{N}$  whose image in  $\mathfrak{g}_{C_\theta^*}/\mathfrak{g}_{F_\theta^*}^+$  is  $e$ . Since  $e$  is a nonzero element of  $\mathfrak{m}$ , it does not lie in the kernel  $\mathfrak{g}_{C_\theta^*}^+/\mathfrak{g}_{F_\theta^*}^+$  of the projection from  $\mathfrak{g}_{C_\theta^*}/\mathfrak{g}_{F_\theta^*}^+$  onto  $\mathfrak{m}$ . In particular, we have that  $X \notin \mathfrak{g}_{C_\theta^*}^+$ . Thus, by Hypothesis 5.3, there exists some  $\mathfrak{sl}_2(\mathfrak{f})$ -triple  $\{f, h, e\}$  in  $V_{C_\theta^*}$  completing  $e$ , and by Corollary 5.14, there exists a lift  $\{Y, H, X\}$  of  $\{f, h, e\}$  which lies in  $\mathfrak{g}_{C_\theta^*}$ . This shows that  $F_\theta^*$  cannot be maximal in  $\mathcal{B}_\theta(Y, H, X)$ .  $\square$

**Lemma 6.21.** *Suppose all hypotheses in Section 5 hold. If  $(F_\theta^*, e) \in I_r^n$  and  $e$  is nontrivial, then  $(F_\theta^*, e) \in I_r^d$  if and only if there exists some  $x \in F_\theta^*$ , a normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple  $(f, h, e) \in V_{x, -r}^- \times V_{x, 0}^+ \times V_{x, r}^-$  completing  $e$ , and a normal  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$  in  $\mathfrak{g}$  lifting  $\{f, h, e\}$  such that  $F_\theta^*$  is a maximal generalized  $(r, \theta)$ -facet of  $\mathcal{B}_\theta(Y, H, X)$ .*

*Proof.* “ $\Rightarrow$ .” This is a consequence of the definition.

“ $\Leftarrow$ .” Let  $x \in F_\theta^*$ ,  $\{f, h, e\}$ , and  $\{Y, H, X\}$  be data satisfying the hypotheses of the lemma such that  $F_\theta^*$  is a maximal generalized  $(r, \theta)$ -facet of  $\mathcal{B}_\theta(Y, H, X)$ . Suppose there is some  $x' \in F_\theta^*$ , a normal  $\mathfrak{sl}_2(\mathfrak{f})$ -triple  $(f', h', e) \in V_{x', -r}^- \times V_{x', 0}^+ \times V_{x', r}^-$  completing  $e$ , and a normal  $\mathfrak{sl}_2(k)$ -triple  $\{Y', H', X'\}$  in  $\mathfrak{g}$  lifting  $\{f', h', e\}$ . We argue by contradiction, and suppose  $F_\theta^*$  is not maximal in  $\mathcal{B}_\theta(Y', H', X')$ . Since  $x, x' \in F_\theta^*$ , we have  $\mathfrak{g}_{x', -r} = \mathfrak{g}_{x, -r}$  and  $\mathfrak{g}_{x, r} = \mathfrak{g}_{x', r}$ . Since  $[\mathfrak{g}_{x', -r}, \mathfrak{g}_{x', r}] \subset \mathfrak{g}_{x, 0}$ , the lift  $(Y', H', X')$  lies in  $\mathfrak{p}_{x, -r} \times \mathfrak{h}_{x, 0} \times \mathfrak{p}_{x, r}$ , so we may assume  $x' = x$ . By Lemma 6.14, we have that  ${}^H X = \mathcal{O}_\theta(F_\theta^*, e) = {}^H X'$ . Combining this with Corollary 6.9,  $X$  is  $H_x^+$ -conjugate to  $X'$ . Thus, by replacing  $\{Y' H', X'\}$  with some  $H_x^+$ -conjugate, we may assume  $X = X'$ . By the last line of Hypothesis 5.9, there exists some element  $h \in C_H(X)$  with  ${}^h Y = Y'$  and  ${}^h H = H'$ . In particular,  $Y \in {}^{h^{-1}}\mathfrak{g}_{x, -r} = \mathfrak{g}_{h^{-1}x, -r}$  and  $X \in {}^{h^{-1}}\mathfrak{g}_{x, r} = \mathfrak{g}_{h^{-1}x, r}$ , so  $h^{-1}\mathcal{B}_\theta(Y', H', X) = \mathcal{B}_\theta(Y, H, X)$ . This shows that  $\mathcal{B}_\theta(Y', H', X)$  and  $\mathcal{B}_\theta(Y, H, X)$  have the same dimension. However, we assumed that  $F_\theta^*$  was not maximal in  $\mathcal{B}(Y', H', X)$ , so  $h^{-1}F_\theta^*$  is not maximal in  $\mathcal{B}_\theta(Y, H, X)$ , which is a contradiction since the action of  $H$  preserves dimension.  $\square$

**Remark 6.22.** Suppose  $(F_{1,\theta}^*, e_1) \sim (F_{2,\theta}^*, e_2)$ . As a consequence of the proof of Lemma 6.21, we have  $(F_{1,\theta}^*, e_1) \in I_r^d$  if and only if  $(F_{2,\theta}^*, e_2) \in I_r^d$ .

**Theorem 6.23.** Suppose all hypotheses of Section 5 hold. There is a bijective correspondence between  $I_r^d / \sim$  and  $\mathcal{O}_\theta(0)$  given by the map that sends  $(F_\theta^*, e)$  to  $\mathcal{O}_\theta(F_\theta^*, e)$ .

*Proof.* We have already shown that the map which sends  $(F_\theta^*, e)$  to  $\mathcal{O}_\theta(F_\theta^*, e)$  is well-defined in Lemma 6.17. We will first show that the map restricted to  $I_r^d$  is injective. Suppose  $\mathcal{O}_\theta(F_{1,\theta}^*, e_1) = \mathcal{O}_\theta(F_{2,\theta}^*, e_2)$ . Recall from Corollary 4.21 that for  $F_\theta^* \in \mathcal{F}(r)$ , the set  $C(F_\theta^*)$  is nonempty. Choose  $x_i \in C(F_{i,\theta}^*)$  and complete  $e_i$  to an  $\mathfrak{sl}_2(\mathfrak{f})$ -triple  $(f_i, h_i, e_i)$  in  $V_{F_{i,\theta}^*}^- \times V_{F_{i,\theta}^*}^+ \times V_{F_{i,\theta}^*}^-$ . By Corollary 5.14, we can lift these  $\mathfrak{sl}_2(\mathfrak{f})$ -triples to  $\mathfrak{sl}_2(k)$ -triples  $\{Y_i, H_i, X_i\}$  in  $\mathfrak{g}$ . By Lemma 6.14,  $\mathcal{O}_\theta(F_{i,\theta}^*, e_i) = {}^H X_i$ . Since  $\mathcal{O}_\theta(F_{1,\theta}^*, e_1) = \mathcal{O}_\theta(F_{2,\theta}^*, e_2)$ , we thus have  ${}^H X_1 = {}^H X_2$ , so by Hypothesis 5.9, there exists an  $h \in H$  such that  $\{Y_1, H_1, X_1\} = \{{}^h Y_2, {}^h H_2, {}^h X_2\}$ . By Lemma 6.14, we have  $F_{1,\theta}^* \subset \mathcal{B}_\theta(Y_1, H_1, X_1)$ , and  $hF_{2,\theta}^* \subset \mathcal{B}_\theta(Y_1, H_1, X_1)$ , so since  $(F_{i,\theta}^*, e_i) \in I_r^d$ ,  $F_{1,\theta}^*$  and  $hF_{2,\theta}^*$  are maximal generalized  $(r, \theta)$ -facets of  $\mathcal{B}_\theta(Y_1, H_1, X_1)$ . Thus, by Lemma 6.6,  $F_{1,\theta}^*$  and  $hF_{2,\theta}^*$  are strongly  $r$ -associated. In particular, there exists an apartment  $\mathcal{A} \subset \mathcal{B}(H)$  such that

$$A(\mathcal{A}, F_{1,\theta}^*) = A(\mathcal{A}, hF_{2,\theta}^*) \neq \emptyset.$$

As  $X_1$  has image  $e_1$  in  $V_{F_{1,\theta}^*}$  and  $X_1$  has image  ${}^h e_2$  in  $V_{hF_{2,\theta}^*}$ , the element  $X_1$  lies in  $\mathfrak{p}_{F_{1,\theta}^*} \cap \mathfrak{p}_{hF_{2,\theta}^*}$ , and

$$e_1 = {}^h e_2 \text{ in } V_{F_{1,\theta}^*} = V_{hF_{2,\theta}^*}$$

Thus, in particular,  $(F_{1,\theta}^*, e_1) \sim (F_{2,\theta}^*, e_2)$ , i.e the map is injective.

For surjectivity, first let  $\{0\}$  be the trivial orbit. Let  $F_\theta^*$  be an open generalized  $(r, \theta)$ -facet, and let  $e$  be the trivial element of  $V_{F_\theta^*}^-$ . Then,  $(F_\theta^*, e)$  maps to  $\{0\}$ . Now, suppose  $\mathcal{O}$  is nontrivial and let  $X \in \mathcal{O}$ . Complete  $X$  to an  $\mathfrak{sl}_2(k)$ -triple  $\{Y, H, X\}$  and choose a maximal

generalized  $(r, \theta)$ -facet  $F_\theta^* \subset \mathcal{B}_\theta(Y, H, X)$ . Let  $e$  denote the image of  $X$  in  $V_{F_\theta^*}$ . Then by Lemma 6.14 (2), we have  $\mathcal{O}_\theta(F_\theta^*, e) = {}^H X$ . □

## 7 Appendix A

### 7.1 Calculation of nilpotent orbits associated to the pair $(\mathbf{SL}_3, \mathbf{PGL}_2)$ .

Recall that we have an involution  $\theta : \mathbf{SL}_3 \rightarrow \mathbf{SL}_3$  defined by  $A \mapsto J(A^t)^{-1}J$ , where  $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ . The Lie algebra of  $\mathbf{SL}_3$ , denoted  $\mathfrak{sl}_3$ , has a decomposition on the level of  $k$ -points which is given by

$$\mathfrak{sl}_3(k) = \begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix} \oplus \begin{pmatrix} x & y & s \\ z & -2x & y \\ u & z & x \end{pmatrix},$$

with  $a, b, c, s, u, x, y, z \in k$ . Let  $X = \begin{pmatrix} x & y & s \\ z & -2x & y \\ u & z & x \end{pmatrix}$  be nilpotent. We would like to find a nicer representative for  $X$  up to  $H = \mathbf{PGL}_2(k)$ -conjugacy.

By Remark 5.5, since  $X$  is nilpotent, there is a one-parameter subgroup  $\lambda : \mathbf{GL}_1 \rightarrow \mathbf{H}$  such that  $\lambda(t)X = t^2X$ . On the other hand, since the image of  $\lambda$  lies in a maximal  $k$ -split torus of  $\mathbf{H}$ , there is some  $h \in \mathbf{H}$  and  $n \in \mathbb{Z}$  such that  $({}^h\lambda)(t) = \begin{pmatrix} t^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-n} \end{pmatrix}$ . Thus, we

have  $({}^{(h\lambda)(t)}({}^hX)) = t^2({}^hX)$ . Letting  $Z = {}^hX = \begin{pmatrix} u & v & w \\ r & -2u & v \\ s & r & u \end{pmatrix}$ , we have

$$\begin{pmatrix} u & t^n v & t^{2n} w \\ t^{-n} r & -2u & t^n v \\ t^{-2n} s & t^{-n} r & u \end{pmatrix} = \begin{pmatrix} t^2 u & t^2 v & t^2 w \\ t^2 r & -2ut^2 & t^2 v \\ t^2 s & t^2 r & t^2 u \end{pmatrix}.$$

Assume  $X$  is nontrivial. If  $v \neq 0$ , we must have  $n = 2$ , but this forces all other entries to be zero. If  $w \neq 0$ , then,  $n$  must be equal to 1, but this forces all other entries to be zero. Thus, the nilpotent  $H$ -conjugacy classes in the  $(-1)$ -eigenspace of  $\mathfrak{sl}_3(k)$  under  $d\theta$  lie in one of the three subsets of  $\mathfrak{sl}_3(k)$ :

$$\{\text{triv}\}, \left\{ \begin{pmatrix} 0 & v & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \mid v \in \mathbb{Q}_p^\times \right\}, \text{ and } \left\{ \begin{pmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid w \in \mathbb{Q}_p^\times \right\},$$

where triv denotes the trivial orbit.

We note that all matrices of the form  $\begin{pmatrix} 0 & v & 0 \\ 0 & 0 & v \\ 0 & 0 & 0 \end{pmatrix}$  are  $\mathbf{H}(k)$ -conjugate by the diagonal

maximal  $\mathbb{Q}_p$ -split torus. Matrices of the form  $\begin{pmatrix} 0 & 0 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  split up into four conjugacy classes which are parametrized by  $(\mathbb{Q}_p)^\times/(\mathbb{Q}_p^\times)^2$ .

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